2026 USAAAO Third Exam (Theory)

1 Instructions (Please Read Carefully)

The test must be completed within 3.5 hours (210 minutes) and the maximum number of points is **260 points**. The point breakdown per problem is as follows.

Problem	P1	P2	P3	P4	P5	**Total**
Points	10	20	50	60	120	**260**

Please solve each problem on a blank piece of paper and mark the number of the problem at the top of the page. Some problems may require a designated answer sheet; these will be provided. The contestant's full name in capital letters should appear at the top of each solution page. If the contestant uses scratch papers, those should be labeled with the contestant's name as well and marked as "scratch paper" at the top of the page. Scratch paper will not be graded. Partial credit will be available given that correct and legible work was displayed in the solution.

This is a written exam. Contestants can only use a scientific calculator (non-programmable and non-graphing) for this exam. A table of physical constants will be provided. **Discussing the problems with other people is strictly prohibited in any way until the end of the examination period on June 25, 2026.** Receiving any external help during the exam is strictly prohibited. This means that the only allowed items are: a calculator, the provided table of constants, a pencil (or pen), an eraser, blank sheets of papers, and the exam. No books or notes are allowed during the exam. Students must be proctored during the entire duration of the exam.

This exam sheet and your answer sheets (including scratch papers) should be returned to your proctors once the exam ended. Your proctors should upload your answer sheets to the Google Form provided to them.

We acknowledge the following people for their contributions to this year's theory exam:

Feodor Yevtushenko, Hagan Hensley, Joe McCarty, Lucas Pinheiro

After reading the instructions, please make sure to sign, affirming that:

1. All work on this exam has been completed by me.
2. I took this exam under the supervision of a proctor.
3. I did not receive any external help beyond the materials provided.
4. I will not discuss the contents of this exam with anyone until June 24, 2026.
5. Failure to follow these rules will result in disqualification from the exam.

Student Signature: ______________________ Date: ______________________

1. **(10 points)**

   For both parts of this question, assume Earth's current orbit is circular with a radius $a = 1$ AU.

   (a) **(3 points)** If the Sun were to instantly lose 40% of its mass, what would happen to Earth's orbit? Assume that the mass simply disappears without interacting with the Earth. Find the semi-major axis a and the eccentricity e of the new orbit.

   (b) **(7 points)** Instead consider the scenario where this same 40% mass loss happens gradually over millions of years, as it will when the Sun is near the end of its life. In this scenario, find the semi-major axis a and the eccentricity e of the new orbit.

2. **(20 points)** *Dubhe*

   On J2000.0, Dubhe's (α UMa) equatorial coordinates were the following: $\delta = 61°45'$ and $\alpha = 11\text{h}04\text{m}$.

   For this problem, neglect Earth's nutation and Dubhe's proper motion.

   (a) **(15 points)** Determine Dubhe's declination on J5000.0.

   (b) **(5 points)** Determine Dubhe's right ascension on J5000.0.

3. **(50 points)** *Recession of the Moon*

   The Apollo missions placed retroreflectors[1] on the surface of the Moon, which has enabled precise measurement of the distance between the Earth and Moon, called *lunar laser ranging* experiments. These experiments determined that the Moon's semi-major axis is increasing at a rate of 38 mm per year. This problem will explore why this happens and what it can teach us about the Earth-Moon system.

   Consider a simplified model where the Earth-Moon system is isolated (i.e., ignore any gravitational influence of the Sun and other planets), the Moon can be treated as a point mass, and the Moon's orbit is circular, with a as the distance between the centers of the Earth and Moon. Let m and $M_\oplus$ be the masses of the Moon and Earth, respectively. Although the center of the Earth revolves around the center of mass of the Earth-Moon system, neglect that motion and treat the center of the Earth as stationary[2]. Let $\Omega = 2\pi/(1 \text{ sidereal day})$ and $\omega = 2\pi/(1 \text{ sidereal month})$ be the angular velocities of the Earth's rotation and the Moon's revolution, respectively. A day is shorter than a month, so $\Omega > \omega$.

   (Throughout this problem, dots denote time derivatives: $\dot{f} = \mathrm{d}f/\mathrm{d}t = f'(t)$.)

   (a) **(2 points)** What is the relationship between ω and a?

   (b) **(5 points)** Find the total angular momentum of the system in terms of Ω, $M_\oplus$, m, a, and the moment of inertia of the Earth about its rotational axis, $I_\oplus$, which you can assume is constant. Since the Moon is moving away from the Earth, both Ω and a are functions of time. What is the relationship between the time derivatives $\dot{\Omega}$ and $\dot{a}$?

   (c) **(7 points)** What is the mechanical energy E (kinetic plus potential) of the system? We are treating the Moon as a point mass, so you can neglect its rotational kinetic energy. For this part, approximate the Earth as spherical for the purpose of calculating the gravitational potential energy of the Moon. Find $\dot{E}$ in terms of m, a, $\dot{a}$, Ω, and ω.

   The process which causes a loss of mechanical energy in this system is called *tidal dissipation*, which is what drove the Moon towards a synchronous orbit around the Earth and is slowly driving the Earth towards being mutually tidal locked with the Moon (which would happen when $\Omega = \omega$). The tidal force of the Moon on the Earth results in a tidal bulge, but because the Earth rotates faster than the Moon revolves, friction drags the bulge so that it is an angle ϵ ahead of the Moon (see Fig. 1)[3]. The bulge then exerts a torque on the Moon, which causes it to recede.

[1] Retroreflectors are an arrangement of mirrors designed to reflect incident light back in the direction from which it came.

[2] This assumption can be made more precise by using the reduced mass, but we won't worry about that for the sake of this problem. The corrections to the recession rate end up being $\mathcal{O}\left((m/M_\oplus)^2\right)$.

[3] In reality, there are separate bulges for the oceans and for the solid Earth, but we will treat it with a single, effective ϵ.

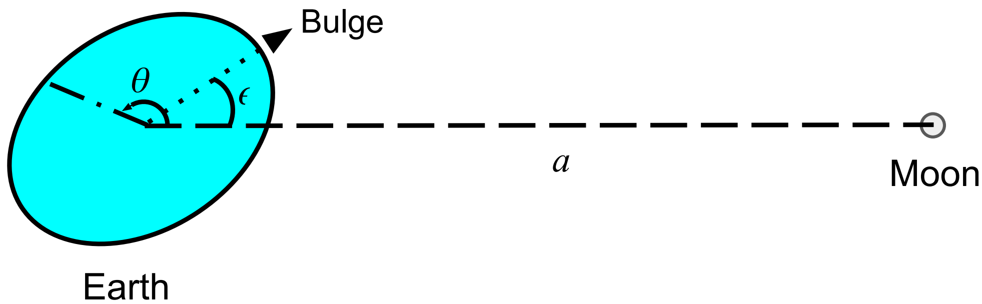


Figure 1: Schematic of the tidal interaction between the Earth and Moon. Not to scale.

(d) **(6 points)** Let τ be the torque exerted by the Earth's bulge on the Moon. In terms of τ, Ω, and ω, what is the total power $\dot{E}$ resulting from the torque of the Earth on the Moon and that of the Moon on the Earth?

(e) **(7 points)** Working in the Earth-Moon plane with the center of the Earth as the origin, and approximating the Earth as a sphere with radius $R_\oplus \ll a$, expand the Moon's gravitational potential to find the *tidal potential* $V_{☾}^{(tidal)}(\theta)$ at the surface of the Earth, as a function of the angle between the Moon and a point on the Earth's surface (see Fig. 1). Note that a constant potential has no effect, and a term proportional to $\sin\theta$ or $\cos\theta$ would produce a constant force (because such terms would be linear potentials in Cartesian coordinates), so your tidal potential $V_{☾}^{(tidal)}(\theta)$ should only contain the term(s) in the expansion proportional to $\sin^2\theta$ and/or $\cos^2\theta$ that are lowest order in the small quantity $R_\oplus/a$.

In order to calculate this torque, we need to know the gravitational effect of Earth's tidal bulge. The potential produced by the Earth in response to the applied tidal potential $V_{☾}(\theta)$, as a function of distance r from the Earth, is

$$V_\oplus(r,\theta) = k_\oplus \left(\frac{R_\oplus}{r}\right)^3 V_{☾}^{(tidal)}(\theta-\epsilon), \tag{1}$$

where $k_\oplus$ is a constant called a *Love number*, which depends on the internal structure and mechanical properties of the Earth. This results in a torque on the Moon of

$$\tau = aF_\perp = -am\left(\frac{1}{a}\frac{\mathrm{d}V_\oplus}{\mathrm{d}\theta}\bigg|_{r=a,\theta=0}\right). \tag{2}$$

(f) **(8 points)** Find an expression for $\dot{a}$ in terms of $k_\oplus$, ϵ, a, and constants.

(g) **(6 points)** Based on the current $\dot{a} = 38\,\mathrm{mm/yr}$, calculate the angle ϵ between the Earth's tidal bulge and the Moon. Take $k_\oplus = 0.3$.

(h) **(9 points)** Suppose the Moon formed with $a \approx 0$. Using the result of part (e), estimate the age of the Moon in years, assuming $k_\oplus$ and ϵ are constant over time. The assumption of constant $k_\oplus$ is probably reasonable, but ϵ may change significantly over geologic time as the tidal dissipation can vary with factors like configuration of continents and the possibility of a "snowball Earth". Based on this, do you think that ϵ was larger, smaller, or about the same in the distant past?

4. **(60 points)** *Stellar Odyssey*

You would like to travel to potentially habitable exoplanet Kepler 22b, located 640 light-years away from Earth. Thankfully, your spaceship is powered by "sufficiently advanced technology" and can accelerate at 1 g (9.81 m/s^2) indefinitely. You plan to accelerate at 1 g until the halfway point of the trip, then turn around and decelerate at 1 g the rest of the way. Assume any relative velocity of Earth and Kepler 22b is negligible, and ignore gravity.

First, let's approach this problem with classical mechanics.

(a) **(5 points)** According to classical mechanics, how long will the trip take and what is the maximum velocity you will reach during the trip?

Based on these answers, is classical mechanics appropriate for describing this situation?

Now let's tackle the problem using special relativity.

Special Relativity Refresher:

A *Lorentz boost* is an origin-preserving coordinate transformation between reference frames moving with relative velocity v. To boost to a frame moving with velocity v along the $+\hat{x}$ direction, we transform space and time coordinates as follows:

$$ct' = \gamma\,(ct - \beta x)$$
$$x' = \gamma\,(x - \beta ct)$$
$$y' = y$$
$$z' = z$$

where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ is the usual Lorentz factor.

Unlike in classical mechanics, the value of a particle's acceleration depends on the reference frame. The *proper acceleration* of a particle is defined to be the acceleration that it experiences in its own (instantaneous) rest frame.

Some identities that might be helpful in this problem:

$$\frac{d}{dx}\arctan(x) = \frac{1}{x^2+1}$$

$$\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$$

$$\int \frac{1}{\sqrt{1+x^2}}dx = \sinh^{-1}(x) + C$$

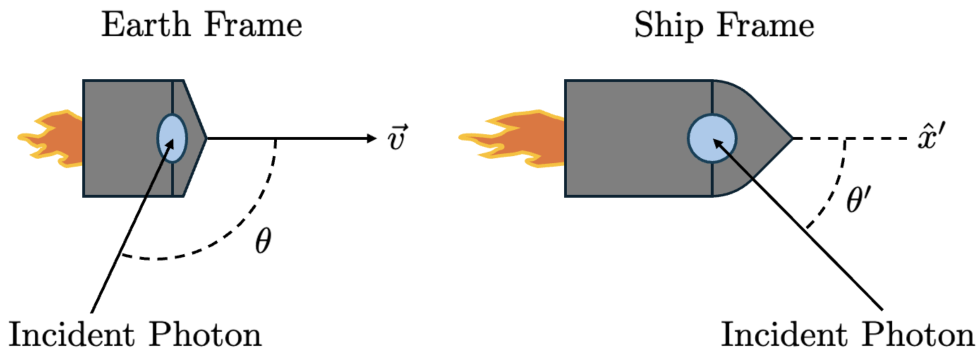
(b) **(14 points)** First, find an expression for the spacetime trajectory (or *worldline*) $x(t)$ for a massive object undergoing constant proper acceleration a, with initial condition $x(0) = v(0) = 0$. For simplicity, consider one-dimensional motion along the $\hat{x}$ direction only, and assume the object is a point mass.

Hint: You may be tempted to derive an expression relating acceleration to proper acceleration, find the acceleration in the Earth frame as a function of the ship's velocity, and integrate the resulting differential equation. This method will give the right answer, but there are more elegant approaches that involve much less calculus!

(c) **(4 points)** How long does the trip take in the Earth's reference frame? Verify that this is longer than the light travel time of 640 years.

(d) **(4 points)** What is the maximum velocity you will reach during the trip (in the Earth's reference frame)? Since this is very close to c, alternatively you can give the maximum value of γ rather than stating v.

(e) **(6 points)** How long will the trip take (proper time) from the ship's perspective?



At various times during the trip, you plan to admire the view out the ship's windows. Let's try to understand what you should expect to see when moving at highly relativistic velocities.

(f) **(10 points)**

Consider a photon hitting the ship with wavelength λ and approaching from angle θ relative to the ship's direction of travel $+\hat{x}$ (λ and θ are defined in Earth's reference frame.) $\theta = 0$ is defined to be hitting the ship from the the $+\hat{x}$ direction (therefore the photon is traveling in the $-\hat{x}$ direction.) If the ship is moving with velocity $v = \beta c$, what are the photon's observed wavelength λ' and angle θ' in the ship's reference frame?

Write your answer in terms of λ, θ, and β (and/or γ.)

(g) **(14 points)**

As you are watching out the side window, you see a planet pass by. You naively expect it to appear squished along the $\hat{x}$ direction due to relativistic length contraction. What shape does the planet actually appear to be? Prove your answer **quantitatively** using the result of (f).

Hint: A useful intermediate step is to show that

$$\frac{\sin\theta'}{\sin\theta} = \frac{d\theta'}{d\theta}$$

(h) **(3 points)**

Based on the result of (f), describe in words what you would see when looking out the front-facing window at the halfway point of the trip. Consider how different sources of light will be affected by the Lorentz boost.

Do you think looking out the front-facing window is a good idea?

5. **(120 points)** *His Dark Materials*

The Cosmic Microwave Background (CMB), first discovered in 1964, is a near-uniform microwave signal from outer space, and it is now widely accepted as being the thermal blackbody radiation from the early Universe.

One of the greatest joint predictions of the Standard Model of particle physics and of modern cosmology is the existence of an analogous *Cosmic Neutrino Background*, or CNB. While the CNB has never been directly observed, its existence is nearly unambiguous in light of our modern understanding of physics. One of the most shocking things about the CNB is that, if it exists, it is possible to derive the ratio of its energy density $\rho_\nu c^2$ to the energy density $\rho_\gamma c^2$ of the CMB solely through a combination of particle physics, thermodynamics, and cosmology. Consequently, using modern CMB measurements, one can calculate the energy density of the CNB. In this problem, you will do exactly this.

(Note: This problem consists of two parts. Part A explores the underlying particle physics and thermodynamics and is worth more points, while Part B applies these results to cosmology and is shorter. In

particular, these parts are ordered by content, not difficulty, and can be attempted in either order (they are independent). We strongly encourage you to try as much as you can.)

Part A: Particle in a Box (90 points)

We will begin by establishing the key underlying thermal physics for how the energy of a system of particles depends on their temperature. Below, we list observed particle properties from Earth-based particle physics experiments (and their corresponding decoupling or annihilation temperatures according to our cosmological and quantum models).

Particle	**Symbol**	**Mass**	g_{anti}	g_{spin}	N_{flavor}	**Type**	**Decouple/Annihilate Temp**
Electron	e^+/e^-	m_e	2	2	1	Fermion	$\approx 5 \times 10^9$ K
Photon	γ	0	1	2	1	Boson	≈ 3000 K
Neutrino	ν	≈ 0	2	1	3	Fermion	$\approx 10^{10}$ K
All Others	–	–	–	–	–	–	$> 10^{12}$ K

Note that while experiments such as neutrino oscillations place a nonzero lower bound on the neutrino mass, we assume it is negligibly small, so treat neutrinos as massless throughout the problem. The antiparticle of the electron is the positron, and the two have the same mass. You may also find the integrals

$$\int_0^\infty \frac{x^n}{e^x - 1}dx = n!\ \zeta(n+1) \text{ and } \int_0^\infty \frac{x^n}{e^x + 1}dx = n!\ (1 - \frac{1}{2^n})\ \zeta(n+1)$$

helpful, where $\zeta(2) = \pi^2/6$, $\zeta(3) \approx 1.202$, and $\zeta(4) = \pi^4/90$. $n!$, or n factorial, denotes the product of the integers from 1 to n, inclusive. The identities

$$|r| < 1 \rightarrow \sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \text{ and } \sum_{k=0}^{\infty} kr^k = \frac{r}{(1-r)^2}$$

may also prove useful.

In the Standard Model of particle physics, energy modes associated with each particle type have a certain number of degrees of freedom g, which may arise due to a particle being distinct from its antiparticle (so positrons count as additional "degrees of freedom" for electrons), it having more than one observed spin state, or it arising in several flavors. Thus, to calculate the total energy stored in all modes for a given particle, it suffices to calculate the energy in one mode and multiply by the number of modes g.

(a) **(2 points)** Based on the above table, propose and justify a chronological ordering of the four events below in the history of the Universe. You must give valid reasoning for credit.

  (i) The overwhelming majority of electrons and positrons pairwise annihilate each other.
  (ii) Photons decouple from ordinary matter and form the Cosmic Microwave Background.
  (iii) Neutrinos decouple from ordinary matter and form the Cosmic Neutrino Background.
  (iv) All other particles either decouple or annihilate.

(b) **(2 points)** For each (named) particle in the above table, calculate g by multiplying the three contributions to g in the above table (including N_{flavor}). Comment on any practical slight deviations to any of the N values, give an estimate, and briefly explain why.

(c) **(1 point)** Later in this problem, we will derive that each fermionic degree of freedom contributes $\frac{7}{8}$ of the energy density as each bosonic degree of freedom; adding this multiplicative correction gives us a new count $g^\star$. At sufficiently early times (just after the Big Bang), none of the Standard Model particles, such as quarks, had yet decoupled or annihilated, making all degrees of freedom active. Estimate the total effective number of degrees of freedom $g^\star_{tot}$ at this point due to all Standard Model particles.

Throughout the rest of this problem, use per-particle g and not $g^\star$.

We will now begin by exploring the basic underlying principles from thermodynamics and statistical mechanics that we will use in this problem. In statistical mechanics, an open, macroscopic system that is in both thermal (exchange of heat) and chemical (exchange of particles) equilibrium with its environment is known as a **grand canonical ensemble**. For now, we only consider identical particles of a given particle type. In such a system, the chemical potential μ denotes the increase in internal energy associated with adding one extra particle to the **environment** (not the system). In other words, transferring a particle from the environment to a state of energy ε in the system has a corresponding total energy change $\varepsilon - \mu$. For a system at temperature T, we write $\beta = 1/(k_B T)$ for convenience. Together, the parameters β and μ characterize the behavior of our system.

First, consider a simple system where particles in the system can only be in one state of energy ε. Relative to a system with zero particles in the system, a state with n particles has energy $E_n = n\varepsilon$. According to statistical mechanics, the probability of a system being in a state with energy E is proportional to $e^{-\beta(E-n\mu)}$. We define the **grand partition function** Ξ as the sum of $e^{-\beta(E-n\mu)}$ over all states, so here we have,

$$\Xi = \sum_n e^{-\beta n(\varepsilon-\mu)},$$

where n runs over all possible numbers of particles in this system. In general, we can use this to express the probability of a system being in a state with energy E and n particles as $e^{-\beta(E-n\mu)}/\Xi$.

(d) **(5 points)** Fermions are particles with half-integer spin. According to the Pauli exclusion principle, no two identical fermions can occupy the same quantum state, so $n \in \{0, 1\}$ above.

  (i) Write out the expression for Ξ for the above system if the particle in question is a fermion. Give the probabilities of the system being in the $n = 0$ and $n = 1$ states.

  (ii) Use this to calculate the expected value (average value) of the number of particles in the system $\langle n_\varepsilon \rangle$ and the expected energy of the system $\langle E_\varepsilon \rangle$.

(e) **(5 points)** Bosons are particles with integer spin and face no such restrictions. As such, $n \geqslant 0$ can be any nonnegative integer.

  (i) Write out and simplify the expression for Ξ for the above system if the particle in question is a boson. Give the probability of the system being in the state with n particles in terms of n.

  (ii) Use this to calculate the expected number of particles in the system $\langle n_\varepsilon \rangle$ and the expected energy of the system $\langle E_\varepsilon \rangle$.

Now consider a system with many individual possible states for particles to occupy. A given state with energy ε has the above probability distribution for its occupancy, independent of all other states. For a macroscopic system, we can simply take the overall energy and number of particles to be their expected values, giving

$$E = \sum_\varepsilon \varepsilon \langle n_\varepsilon \rangle \text{ and } N = \sum_\varepsilon \langle n_\varepsilon \rangle,$$

where the sum is over all states (indexed by their energies ε). Partition functions over independent states multiply, so

$$\Xi = \prod_\varepsilon \Xi_\varepsilon,$$

where Ξ_ε is the partition function for that state alone (independently of all others).

(f) **(5 points)** Verify the identity

$$E = -\left(\frac{\partial \ln \Xi}{\partial \beta}\right)_\mu + \mu N,$$

holds for both the fermionic and bosonic cases, where, as usual, the subscript to the bottom-right of a partial derivative denotes keeping that variable constant while taking the partial derivative.

We will now consider a macroscopic system of identical particles in a grand canonical ensemble of a known V, μ, and β. We will make the following assumptions about the system:

- Particles do not interact with each other. (Note that this does *not* mean that the states of all particles are independent, as principles like the Pauli exclusion principle may still apply. Rather, particles do not exert forces on each other and thus may be treated with the framework we establish.)
- The system is homogeneous and isotropic, so the energy of a particle ε solely depends on the magnitude of its momentum p.

In particular, we assume that there is no external potential, meaning that we have a simple relationship between energy and momentum.

(g) **(5 points)** Consider a particle of mass m (possibly equal to zero) and momentum p. Give your answers to the below questions in terms of m, p, and physical constants.

  (i) According to relativity, what is the **general** expression for ε in terms of the above variables that is valid for all speeds?

  (ii) Simplify your answer above if the particle is either massless or ultrarelativistic ($p \gg mc$).

  (iii) Instead, assume the particle is massive and not relativistic ($p \ll mc$). Give the Taylor expansion of the formula for ε to leading nonzero order in p.

In the below subparts, we will denote the energy of a state as being a function of its momentum via $\varepsilon(p)$. Now, according to quantum statistics, the number of single particle states in a given volume d^3p of 3D momentum space (the state of all possible momenta of a particle) of a system with volume V and degeneracy g is $gV/(2\pi\hbar)^3 \cdot d^3p$. Note that integrating over all of momentum space is equivalent to integrating each dimension of momentum (p_x, p_y, and p_z) from negative infinity to infinity. *(It may be helpful that in an isotropic system, this is equivalent to integrating $4\pi^2 dp$ from zero to infinity.)*

(h) **(10 points)** Write the total energy E and number of particles N as integrals over the **magnitude** of the momentum p. Express your answers in terms of g, V, $\hbar$, β, μ, and the general functional form $\varepsilon(p)$. **You do not need to evaluate the integrals.** *(You may either do this separately for both fermions and bosons or do them together with a cleverly placed $\pm$ sign. If you do the latter, specify which branch corresponds to what particle type.)*

Now, in the early Universe, photons could be created and destroyed freely in various events, so their chemical potential was $\mu_\gamma = 0$.

(i) **(10 points)** Calculate the energy density of a photon gas $u_\gamma = E_\gamma/V$ in terms of its temperature T and fundamental constants. State how the energy density scales with temperature.

(j) **(5 points)** Repeat this calculation for a massless or ultrarelativistic fermion gas of temperature T and degrees of freedom g, again assuming that the chemical potential is zero. Substitute the values of g for both electrons and neutrinos.

We will now turn our attention to the entropy S of the system. Letting P be the pressure of the system, define the grand potential Ω as

$$\Omega = -k_B T \ln \Xi = E - TS - \mu N = -PV.$$

Also recall the differential form of the first law of thermodynamics:

$$dE = TdS - PdV + \mu dN$$

You may freely use all the above equations without proof. Be careful to not omit (relevant) terms when working with differentials.

(k) **(5 points)** Prove the identity

$$S = -\left(\frac{\partial \Omega}{\partial T}\right)_{V,\mu},$$

where we keep the volume and chemical potential constant.

(l) **(10 points)** Using the same integral form as earlier, express S as an integral over p using V, β, μ, $\varepsilon(p)$, and fundamental constants. Be sure to handle both the fermion and boson cases.

(m) **(15 points)** Now, consider only massless or ultrarelativistic particles. Express the energy E in terms of the pressure P and the volume V. *(It may be helpful to handle fermions and bosons separately and use integration by parts on an integral expression for* $\ln \Xi$*. Alternatively, you may also choose to use a more classical thermodynamic argument to directly obtain an expression for both cases simultaneously. Either way, you* ***must*** *give a complete derivation for credit.)*

(n) **(10 points)** Use this to write the entropy density $s = S/V$ in terms of the energy density $u = E/V$, the number density $n = N/V$, μ, and T for massless or ultrarelativistic particles. Give a second simplified answer for the case $\mu = 0$.

Part B: Nothing, Except Everything (30 points)

As the Universe continued to expand, after neutrinos decoupled from ordinary matter, electrons and positrons became less relativistic, after which virtually all electrons and positrons annihilated each other. Since photons had not yet decoupled from ordinary matter, the energy from the electrons and positrons went into heating the photon gas, but it did not heat the neutrino background. As a result, while neutrinos and photons were at the same temperature before the electron annihilation, the photon gas became warmer than the neutrino background. We will now compute the corresponding temperature and energy density ratios of the present-day photon and neutrino backgrounds.

You may use the results $u \propto gg'T^4$ (energy density) and $s \propto gg'T^3$ (entropy density) for ultrarelativistic particles from the previous section throughout this section. Here, g is the number of degrees of freedom from (b) and g' is 1 if the particle is a boson or 7/8 if it is a fermion.

Assume a smooth, uniform expanding universe in equilibrium with itself. Consider a box of volume V that expands with the Universe (it is comoving), so $V \propto a^3$, where a is the scale factor. (Specifically, any distance d in the present day corresponds to two points that were separated by a distance ad in the past at a point where the scale factor was a, after which the points moved apart due to the expansion of the Universe.)

(o) **(5 points)** First of all, we will show that the total energy in the box is likely *not* conserved. To do this, consider a universe made out of only a photon gas and state the proportionality relations for how T and $U = uV$ scale with a.

(p) **(10 points)** On the other hand, under these assumptions, the total entropy in the comoving box *is* conserved. Indeed, according to the First Law of Thermodynamics,

$$dU = -PdV + TdS \rightarrow d(uV) = -PdV + TdS.$$

Use this in conjunction with the Friedmann continuity equation

$$\dot{\rho} + 3H(\rho + \frac{P}{c^2}) = 0$$

to show that $dS = 0$ over a differential timestep dt.

Before neutrino decoupling, all three relevant types of particles (electrons, neutrinos, and photons) were ultrarelativistic and were at the same temperature. After neutrino decoupling, the comoving entropies of both the neutrino system and the electron plus photon system stayed constant. In particular,

$$\frac{s_e a^3 + s_\gamma a^3}{s_\nu a^3} = \text{const.}$$

(q) **(10 points)** By evaluating the above both before electron annihilation (while electrons were still ultrarelativistic) and after electron annihilation (where there were essentially no electrons), calculate the present-day photon to neutrino temperature ratio T_γ/T_ν. Evaluate T_ν to three significant figures given $T_\gamma \approx 2.725$ K.

(r) **(5 points)** Calculate the energy density ratio ρ_ν/ρ_γ and evaluate this to three significant figures.