2026 USAAAO Third Exam (Theory)

1 Instructions (Please Read Carefully)

The test must be completed within 3.5 hours (210 minutes) and the maximum number of points is **260 points**. The point breakdown per problem is as follows.

Problem	P1	P2	P3	P4	P5	**Total**
Points	10	20	50	60	120	**260**

Please solve each problem on a blank piece of paper and mark the number of the problem at the top of the page. Some problems may require a designated answer sheet; these will be provided. The contestant's full name in capital letters should appear at the top of each solution page. If the contestant uses scratch papers, those should be labeled with the contestant's name as well and marked as "scratch paper" at the top of the page. Scratch paper will not be graded. Partial credit will be available given that correct and legible work was displayed in the solution.

This is a written exam. Contestants can only use a scientific calculator (non-programmable and non-graphing) for this exam. A table of physical constants will be provided. **Discussing the problems with other people is strictly prohibited in any way until the end of the examination period on June 25, 2026.** Receiving any external help during the exam is strictly prohibited. This means that the only allowed items are: a calculator, the provided table of constants, a pencil (or pen), an eraser, blank sheets of papers, and the exam. No books or notes are allowed during the exam. Students must be proctored during the entire duration of the exam.

This exam sheet and your answer sheets (including scratch papers) should be returned to your proctors once the exam ended. Your proctors should upload your answer sheets to the Google Form provided to them.

We acknowledge the following people for their contributions to this year's theory exam:

Feodor Yevtushenko, Hagan Hensley, Joe McCarty, Lucas Pinheiro

After reading the instructions, please make sure to sign, affirming that:

1. All work on this exam has been completed by me.
2. I took this exam under the supervision of a proctor.
3. I did not receive any external help beyond the materials provided.
4. I will not discuss the contents of this exam with anyone until June 24, 2026.
5. Failure to follow these rules will result in disqualification from the exam.

Student Signature: ______________________________ Date: ____

1. **(10 points)**

For both parts of this question, assume Earth's current orbit is circular with a radius $a = 1$ AU.

(a) **(3 points)** If the Sun were to instantly lose 40% of its mass, what would happen to Earth's orbit? Assume that the mass simply disappears without interacting with the Earth. Find the semi-major axis a and the eccentricity e of the new orbit.

(b) **(7 points)** Instead consider the scenario where this same 40% mass loss happens gradually over millions of years, as it will when the Sun is near the end of its life. In this scenario, find the semi-major axis a and the eccentricity e of the new orbit.

Solution:

(a) The new mass of the Sun is $0.6M_\odot$. The Earth is still moving with its original orbital velocity $v = \sqrt{\frac{GM_\odot}{1\text{ AU}}}$ immediately after the Sun loses its mass, but this is now faster than the velocity required for a circular orbit. Vis-viva gives

$$v^2 = \frac{GM_\odot}{1\text{ AU}} = 0.6GM_\odot\left(\frac{2}{1\text{ AU}} - \frac{1}{a}\right)$$

$$a = \frac{1}{\frac{2}{1\text{ AU}} - \frac{1}{0.6\times 1\text{ AU}}} = \boxed{3\text{ AU}}\ (\mathbf{1.5\ points})$$

The perihelion distance should still be 1 AU, so we also have $e = 1 - \frac{1\text{ AU}}{3\text{ AU}} = \boxed{0.67}$ (**1.5 points**).

(b) This process occurs much more slowly than Earth's orbital period, so there's nothing to break the symmetry of the initially circular orbit as in part (a). Thus, the orbit should remain circular and gradually move outwards as the Sun loses its mass. Since the orbit is circular we have $\boxed{e = 0}$ (**1 point**).

Conservation of energy does not apply here, since this is an open system losing mass to its environment. However, conservation of the Earth's angular momentum should still hold (**2 points** for realizing this). We can justify this more carefully using the (classical) adiabatic theorem, which in this case says that the Earth's angular momentum is an *adiabatic invariant*: it is unchanged for very slow changes in the system Hamiltonian.

If $M_\oplus \ll M_\odot$ and the Earth has a circular orbit with semimajor axis a, then the Earth's angular momentum is

$$L = M_\oplus av = M_\oplus\sqrt{GM_\odot a}\ (\mathbf{1\ point})$$

This quantity must be conserved. Since $M_\oplus$ and G don't change, this implies $M_\odot a$ must be constant as $M_\odot$ drops (**1 point**). Thus we have

$$\frac{a}{1\text{ AU}} = \frac{M_\odot}{0.6M_\odot}$$

$$\boxed{a = 1.67\text{ AU}}\ (\mathbf{2\ points})$$

This process will happen in several billion years when the Sun expands into a red giant and loses much of its mass to solar wind. The Earth's expanded orbit may save it from being swallowed by the red giant Sun, though the exact outcome remains an open question.

2. **(20 points)** *Dubhe*

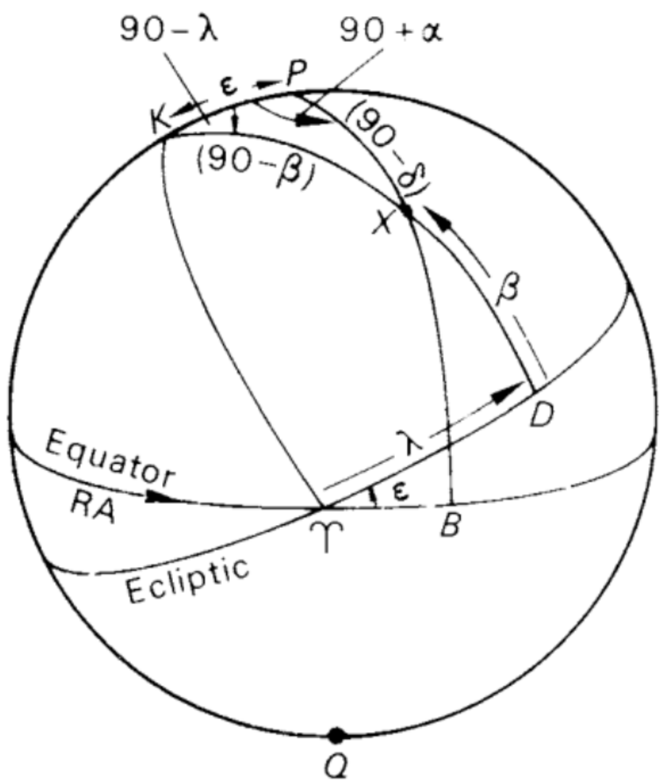
On J2000.0, Dubhe's (α UMa) equatorial coordinates were the following: $\delta = 61°45'$ and $\alpha = 11\text{h}04\text{m}$. For this problem, neglect Earth's nutation and Dubhe's proper motion.

(a) **(15 points)** Determine Dubhe's declination on J5000.0.

(b) **(5 points)** Determine Dubhe's right ascension on J5000.0.

Solution:

(a) The first step is to calculate the ecliptic coordinates of Dubhe. It is possible to obtain the expressions for the coordinate conversions from the PKX triangle in the figure below.



Source of the image: Astronomy Principles and Practice by Roy and Clarke

The ecliptic latitude (β) can be obtained using the spherical law of cosines.

$$
\begin{aligned}
\cos(90° - \beta) &= \cos\varepsilon\cos(90° - \delta) + \sin\varepsilon\sin(90° - \delta)\cos(90° + \alpha) \\
\sin\beta &= \cos\varepsilon\sin\delta - \sin\varepsilon\cos\delta\cos\alpha \\
\sin\beta &= \cos 23.44°\sin 61°45' - \sin 23.44°\cos 61°45'\cos 11\text{h}04\text{m} \\
\sin\beta &= 0.7626 \\
\beta &= 49°42'
\end{aligned}
$$

The ecliptic longitude (λ) can also be obtained from the spherical law of cosines.

$$
\begin{aligned}
\cos(90^\circ - \delta) &= \cos\varepsilon\cos(90^\circ - \beta) + \sin\varepsilon\sin(90^\circ - \beta)\cos(90^\circ - \lambda)\\
\sin\delta &= \cos\varepsilon\sin\beta + \sin\varepsilon\cos\beta\sin\lambda\\
\sin\lambda &= \frac{\sin\delta - \cos\varepsilon\sin\beta}{\sin\varepsilon\cos\beta}\\
\sin\lambda &= \frac{\sin 61^\circ 45' - \cos 23.44^\circ \sin 49^\circ 42'}{\sin 23.44^\circ \cos 49^\circ 42'}\\
\sin\lambda &= 0.7042
\end{aligned}
$$

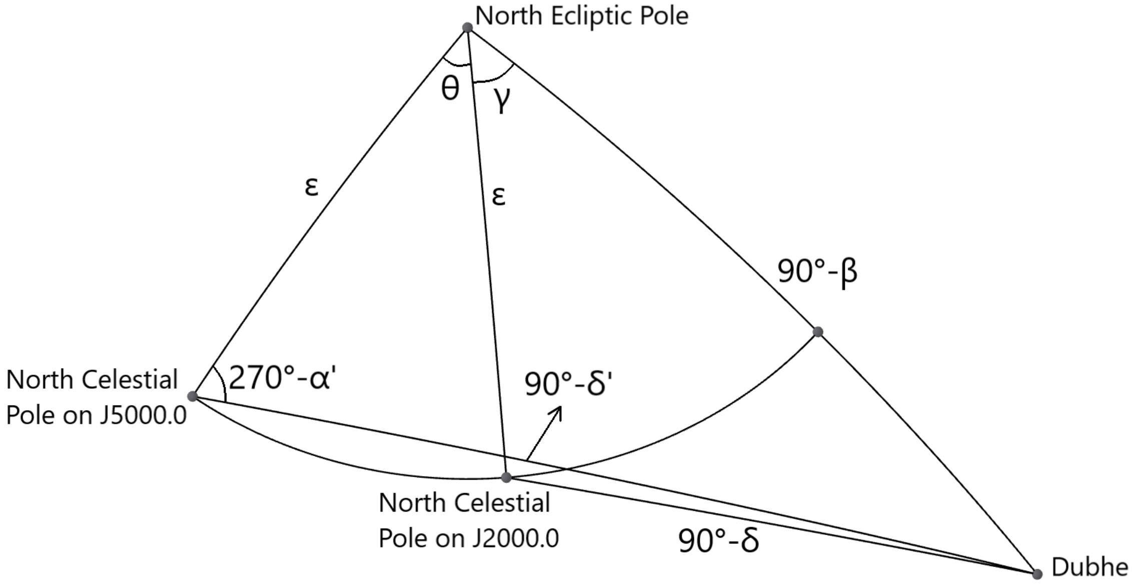
It is important to be careful with an ambiguity here. There are two angles with the same sine, and since the range for λ is $[0^\circ, 360^\circ)$, it is not possible to know which of the angles is the correct one right away. It is possible to resolve this ambiguity with the cosine of λ, which can be obtained with the spherical law of sines.

$$
\begin{aligned}
\frac{\sin(90^\circ - \lambda)}{\sin(90^\circ - \delta)} &= \frac{\sin(90 + \alpha)}{\sin(90^\circ - \beta)}\\
\frac{\cos\lambda}{\cos\delta} &= \frac{\cos\alpha}{\cos\beta}\\
\cos\lambda &= \frac{\cos\delta\cos\alpha}{\cos\beta}
\end{aligned}
$$

It is always true that $\cos\beta > 0$ and $\cos\delta > 0$. Since $\cos 11\text{h}04\text{m} = -0.9703 < 0$, $\cos\lambda < 0$. With a positive sine and a negative cosine, λ must be in the second quadrant.

$$
\lambda = 135^\circ 14'
$$

The ecliptic longitude of the North Celestial Pole is $\lambda_{NCP} = 90^\circ$. Therefore, it is west of Dubhe in the ecliptic system of coordinates. With this in mind, it is possible to draw the figure below.



The equatorial coordinates of Dubhe change throughout time due to Earth's precession. Since the North Celestial Pole moves westward throughout the precession and the pole was already west of Dubhe on J2000.0, the pole will be even further from Dubhe's meridian of ecliptic longitude by J5000.0.

The angle γ is simply the difference in ecliptic longitudes between the North Celestial Pole and Dubhe.

$$\begin{aligned}
\gamma &= \lambda - \lambda_{NCP} \\
\gamma &= 135^\circ 14' - 90^\circ \\
\gamma &= 45^\circ 14'
\end{aligned}$$

Since the rate of precession is constant, it is possible to use the interval between J2000.0 and J5000.0 and the precession period to obtain the value of the angle θ.

$$\begin{aligned}
\theta &= 360^\circ \cdot \frac{3000}{25772} \\
\theta &= 41^\circ 54'
\end{aligned}$$

Note that $\theta + \gamma < 180^\circ$, so the spherical triangle with vertices at Dubhe, the North Ecliptic Pole, and the North Celestial Pole at J5000.0 is a valid spherical triangle.

Applying the law of cosines to this triangle, it is possible to obtain the declination of Dubhe on J5000.0 (δ').

$$\begin{aligned}
\cos(90^\circ - \delta') &= \cos\varepsilon \cos(90^\circ - \beta) + \sin\varepsilon \sin(90^\circ - \beta)\cos(\theta + \gamma) \\
\sin\delta' &= \cos\varepsilon \sin\beta + \sin\varepsilon \cos\beta \cos(\theta + \gamma) \\
\sin\delta' &= \cos 23.44^\circ \sin 49^\circ 42' + \sin 23.44^\circ \cos 49^\circ 42' \cos(41^\circ 54' + 45^\circ 14') \\
\sin\delta' &= 0.7125 \\
&\boxed{\delta' = 45^\circ 26'}
\end{aligned}$$

(b) It is possible to use the law of cosines to obtain the right ascension of Dubhe on J5000.0 (α'). It is worth noting that the internal angle at the North Celestial Pole on J5000.0 corresponds to $270^\circ - \alpha'$ becuse the right ascension of the North Ecliptic Pole corresponds to 270°, and Dubhe is west of the North Ecliptic pole in the equatorial system of coordinates.

$$\begin{aligned}
\cos(90^\circ - \beta) &= \cos\varepsilon \cos(90^\circ - \delta') + \sin\varepsilon \sin(90^\circ - \delta')\cos(270^\circ - \alpha') \\
\sin\beta &= \cos\varepsilon \sin\delta' - \sin\varepsilon \cos\delta' \sin\alpha' \\
\sin\alpha' &= \frac{\cos\varepsilon \sin\delta' - \sin\beta}{\sin\varepsilon \cos\delta'} \\
\sin\alpha' &= \frac{\cos 23.44^\circ \sin 45^\circ 26' - \sin 49^\circ 42'}{\cos 23.44^\circ \cos 45^\circ 26'} \\
\sin\alpha' &= -0.3902
\end{aligned}$$

There are two angles with this sine, but one of them would be over 270° and therefore impossible with this geometry. Therefore, the correct right ascension is the one in the third quadrant.

$$\boxed{\alpha' = 13\text{h}32\text{m}}$$

Rubrics:

- **Item A:**
  - **(2 points)** Determining the ecliptic latitude of Dubhe.
  - **(2 points)** Determining the ecliptic longitude of Dubhe.
  - **(1 points)** Realizing that the ecliptic longitude of the North Ecliptic Pole is $90°$.
  - **(6 points)** Correctly drawing or describing the geometry of the problem.
  - **(1 point)** Determining the value of θ.
  - **(1 point)** Determining the value of γ
  - **(2 points)** Correct final answer.
- **Item B:**
  - **(3 points)** Identifying the angle $270° - \alpha'$ in the diagram.
  - **(2 points)** Correct final answer.

3. **(50 points)** *Recession of the Moon*

The Apollo missions placed retroreflectors[1] on the surface of the Moon, which has enabled precise measurement of the distance between the Earth and Moon, called *lunar laser ranging* experiments. These experiments determined that the Moon's semi-major axis is increasing at a rate of 38 mm per year. This problem will explore why this happens and what it can teach us about the Earth-Moon system.

Consider a simplified model where the Earth-Moon system is isolated (i.e., ignore any gravitational influence of the Sun and other planets), the Moon can be treated as a point mass, and the Moon's orbit is circular, with a as the distance between the centers of the Earth and Moon. Let m and $M_\oplus$ be the masses of the Moon and Earth, respectively. Although the center of the Earth revolves around the center of mass of the Earth-Moon system, neglect that motion and treat the center of the Earth as stationary[2]. Let $\Omega = 2\pi/(1 \text{ sidereal day})$ and $\omega = 2\pi/(1 \text{ sidereal month})$ be the angular velocities of the Earth's rotation and the Moon's revolution, respectively. A day is shorter than a month, so $\Omega > \omega$.

(Throughout this problem, dots denote time derivatives: $\dot{f} = \mathrm{d}f/\mathrm{d}t = f'(t)$.)

(a) **(2 points)** What is the relationship between ω and a?

(b) **(5 points)** Find the total angular momentum of the system in terms of Ω, $M_\oplus$, m, a, and the moment of inertia of the Earth about its rotational axis, $I_\oplus$, which you can assume is constant. Since the Moon is moving away from the Earth, both Ω and a are functions of time. What is the relationship between the time derivatives $\dot{\Omega}$ and $\dot{a}$?

(c) **(7 points)** What is the mechanical energy E (kinetic plus potential) of the system? We are treating the Moon as a point mass, so you can neglect its rotational kinetic energy. For this part, approximate the Earth as spherical for the purpose of calculating the gravitational potential energy of the Moon. Find $\dot{E}$ in terms of m, a, $\dot{a}$, Ω, and ω.

[1] Retroreflectors are an arrangement of mirrors designed to reflect incident light back in the direction from which it came.

[2] This assumption can be made more precise by using the reduced mass, but we won't worry about that for the sake of this problem. The corrections to the recession rate end up being $\mathcal{O}\left((m/M_\oplus)^2\right)$.

The process which causes a loss of mechanical energy in this system is called *tidal dissipation*, which is what drove the Moon towards a synchronous orbit around the Earth and is slowly driving the Earth towards being mutually tidal locked with the Moon (which would happen when $\Omega = \omega$). The tidal force of the Moon on the Earth results in a tidal bulge, but because the Earth rotates faster than the Moon revolves, friction drags the bulge so that it is an angle ϵ ahead of the Moon (see Fig. 1)[3]. The bulge then exerts a torque on the Moon, which causes it to recede.

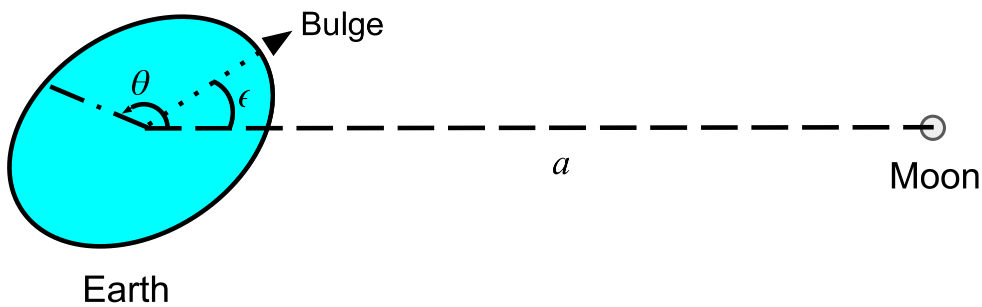


Figure 1: Schematic of the tidal interaction between the Earth and Moon. Not to scale.

(d) **(6 points)** Let τ be the torque exerted by the Earth's bulge on the Moon. In terms of τ, Ω, and ω, what is the total power $\dot{E}$ resulting from the torque of the Earth on the Moon and that of the Moon on the Earth?

(e) **(7 points)** Working in the Earth-Moon plane with the center of the Earth as the origin, and approximating the Earth as a sphere with radius $R_\oplus \ll a$, expand the Moon's gravitational potential to find the *tidal potential* $V_{☾}^{(tidal)}(\theta)$ at the surface of the Earth, as a function of the angle between the Moon and a point on the Earth's surface (see Fig. 1). Note that a constant potential has no effect, and a term proportional to $\sin\theta$ or $\cos\theta$ would produce a constant force (because such terms would be linear potentials in Cartesian coordinates), so your tidal potential $V_{☾}^{(tidal)}(\theta)$ should only contain the term(s) in the expansion proportional to $\sin^2\theta$ and/or $\cos^2\theta$ that are lowest order in the small quantity $R_\oplus/a$.

In order to calculate this torque, we need to know the gravitational effect of Earth's tidal bulge. The potential produced by the Earth in response to the applied tidal potential $V_{☾}(\theta)$, as a function of distance r from the Earth, is

$$V_\oplus(r,\theta) = k_\oplus \left(\frac{R_\oplus}{r}\right)^3 V_{☾}^{(tidal)}(\theta - \epsilon), \tag{1}$$

where $k_\oplus$ is a constant called a *Love number*, which depends on the internal structure and mechanical properties of the Earth. This results in a torque on the Moon of

$$\tau = aF_\perp = -am\left(\frac{1}{a}\frac{\mathrm{d}V_\oplus}{\mathrm{d}\theta}\bigg|_{r=a,\theta=0}\right). \tag{2}$$

(f) **(8 points)** Find an expression for $\dot{a}$ in terms of $k_\oplus$, ϵ, a, and constants.

(g) **(6 points)** Based on the current $\dot{a} = 38\,\mathrm{mm/yr}$, calculate the angle ϵ between the Earth's tidal bulge and the Moon. Take $k_\oplus = 0.3$.

(h) **(9 points)** Suppose the Moon formed with $a \approx 0$. Using the result of part (e), estimate the age of the Moon in years, assuming $k_\oplus$ and ϵ are constant over time. The assumption of constant $k_\oplus$ is

[3] In reality, there are separate bulges for the oceans and for the solid Earth, but we will treat it with a single, effective ϵ.

probably reasonable, but ϵ may change significantly over geologic time as the tidal dissipation can vary with factors like configuration of continents and the possibility of a "snowball Earth". Based on this, do you think that ϵ was larger, smaller, or about the same in the distant past?

Solution: This solution mostly follows Chapter 4 of Murray & Dermott (1999).

(a) By Kepler's Third Law, with $T = 2\pi/\omega$ as the orbital period (and stationary center of the Earth corresponding to $M_\oplus \gg m$), $4\pi^2 a^3 = GM_\oplus T^2$, **(1 point)** so

$$\boxed{\omega^2 a^3 = GM_\oplus}\ \textbf{(1 point)}. \tag{3}$$

No points deducted for using $M_\oplus + m$ instead of just $M_\oplus$.

(b) Angular momentum of the Earth: $I_\oplus \Omega$ **(1 point)**. Orbital angular momentum of the Moon: $ma^2\omega$ **(1 point)**. Combining and using Eq. 3,

$$L = I_\oplus \Omega + ma^2\omega = \boxed{I_\oplus \Omega + m\sqrt{GM_\oplus a}}\ \textbf{(1 point)}. \tag{4}$$

Differentiating and using conservation of angular momentum $\dot{L} = 0$, **(1 point)**

$$\begin{aligned} 0 &= I_\oplus \dot{\Omega} + \frac{m}{2}\sqrt{\frac{GM_\oplus}{a}}\dot{a} \\ \Rightarrow &\boxed{I_\oplus \dot{\Omega} = -\frac{m}{2}\sqrt{\frac{GM_\oplus}{a}}\dot{a}}\ \textbf{(1 point)}. \end{aligned} \tag{5}$$

(c) Rotational kinetic energy of the Earth: $\frac{1}{2}I_\oplus \Omega^2$ **(2 points)**. Total orbital energy of the Moon: $-\frac{GM_\oplus m}{2a}$ **(2 points)**. Combining, **(1 point)**

$$\boxed{E = \frac{1}{2}I_\oplus \Omega^2 - \frac{GM_\oplus m}{2a}}. \tag{6}$$

Differentiating **(1 point)** and inserting $\dot{\Omega}$ from Eq. 4,

$$\begin{aligned} \dot{E} &= I_\oplus \Omega \dot{\Omega} + \frac{GM_\oplus m}{2a^2}\dot{a} = -\frac{m}{2}\sqrt{\frac{GM_\oplus}{a}}\dot{a}\Omega + \frac{GM_\oplus m}{2a^2}\dot{a} \\ &= -\frac{m}{2}\sqrt{\frac{GM_\oplus}{a}}\dot{a}\left(\Omega - \sqrt{\frac{GM_\oplus}{a^3}}\right) = \boxed{-\frac{m}{2}\omega a\dot{a}(\Omega - \omega)}\ \textbf{(1 point)}. \end{aligned} \tag{7}$$

Note that ω and a are related by Eq. 3, so there are multiple accepted answers here. This form is nice because it makes it clear that, if $\dot{E} < 0$ and $\Omega > \omega$, $\dot{a} > 0$ (recession).

(d) The work done by a torque τ as an object rotates by a small angle $\Delta\phi$ is $\tau\Delta\phi$, so the power of the Earth's torque on the Moon is $\tau\omega$ **(2 points)**. By conservation of angular momentum, the Moon exerts the same magnitude of torque back on the Earth but with opposite sign **(2 points)**, so the power of the Moon's torque on the Earth is $-\tau\Omega$ **(1 point)**. The total is

$$\boxed{\dot{E} = -\tau(\Omega - \omega)}\ \textbf{(1 point)}. \tag{8}$$

Note that $\dot{E} < 0$ because $\Omega > \omega$.

(e) The gravitational potential from the Moon at a distance $r_☾$ from the Moon is **(1 point)**

$$V_☾ = -\frac{Gm}{r_☾}.$$

By law of cosines, the distance from the Moon to a point on Earth's surface is **(1 point)**

$$r_☾ = \sqrt{a^2 + R_\oplus^2 - 2aR_\oplus \cos\theta} = a\sqrt{1 + (R_\oplus/a)^2 - 2(R_\oplus/a)\cos\theta}, \tag{9}$$

so

$$V_☾ = -\frac{Gm}{a}\left(1 + (R_\oplus/a)^2 - 2(R_\oplus/a)\cos\theta\right)^{-1/2}. \tag{10}$$

Taylor expanding for small x, **(1 point)**

$$(1+x)^{-1/2} \approx 1 + \frac{\mathrm{d}}{\mathrm{d}x}\left((1+x)^{-1/2}\right)\Big|_{x=0} x + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left((1+x)^{-1/2}\right)\Big|_{x=0} x^2 = 1 - \frac{1}{2}x + \frac{3}{8}x^2. \tag{11}$$

Here, using $x = (R_\oplus/a)^2 - 2(R_\oplus/a)\cos\theta$ **(1 point for realizing what to expand with)**, the only term that is lower than third order in $R_\oplus/a$ that will contain $\cos^2\theta$ comes from x^2 **(1 point)** (a higher order Taylor expansion would include more $\cos^2\theta$ terms, but they would be $\mathcal{O}\left((R_\oplus/a)^3\right)$). Writing down just that term and throwing away the rest,

$$V_☾^{(tidal)}(\theta) = -\frac{Gm}{a}\cdot\frac{3}{8}\left(-2(R_\oplus/a)\cos\theta\right)^2 = \boxed{-\frac{3GmR_\oplus^2}{2a^3}\cos^2(\theta)}. \tag{12}$$

(2 points for final answer; fine if replaced $\cos^2$ with $-\sin^2$ because constant potential terms don't matter)

Doing the Taylor expansion more completely,

$$\begin{aligned} V_☾ &\approx -\frac{Gm}{a}\left(1 - \frac{1}{2}(R_\oplus/a)^2 + (R_\oplus/a)\cos\theta + \frac{3}{8}\left((R_\oplus/a)^2 - 2(R_\oplus/a)\cos\theta\right)^2 + \mathcal{O}\left((R_\oplus/a)^3\right)\right) \\ &\approx -\frac{Gm}{a} - \frac{Gm}{a^2}R_\oplus\cos\theta + \frac{GmR_\oplus^2}{2a^3} - \frac{3GmR_\oplus^2}{2a^3}\cos^2(\theta) + \frac{Gm}{a}\cdot\mathcal{O}\left((R_\oplus/a)^3\right). \end{aligned} \tag{13}$$

The first and third terms are constants. The second terms is proportional to $R_\oplus\cos\theta$, which is a Cartesian coordinate (e.g., z), so this term in the potential just gives the constant gravitational acceleration Gm/a^2. The fourth term is all that we are interested in here, because everything else is higher order in $(R_\oplus/a)$.

(f) Inserting, **(1 point)**

$$V_\oplus(r,\theta) = k_\oplus\left(\frac{R_\oplus}{r}\right)^3 V_☾^{(tidal)}(\theta - \epsilon) = -\frac{3}{2}k_\oplus\frac{GmR_\oplus^5}{a^3r^3}\cos^2(\theta-\epsilon), \tag{14}$$

so

$$\begin{aligned} \tau &= -am\left(\frac{1}{a}\frac{\mathrm{d}V_\oplus}{\mathrm{d}\theta}\Big|_{r=a,\theta=0}\right) = \frac{3}{2}k_\oplus\frac{Gm^2R_\oplus^5}{a^6}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\cos^2(\theta-\epsilon)\right)\Big|_{\theta=0} \\ &= -3k_\oplus\cos(-\epsilon)\sin(-\epsilon)\frac{Gm^2R_\oplus^5}{a^6} \quad \textbf{(2 points for differentiating)} \\ &= 3k_\oplus\cos\epsilon\sin\epsilon\frac{Gm^2R_\oplus^5}{a^6} = \frac{3}{2}k_\oplus\sin(2\epsilon)\frac{Gm^2R_\oplus^5}{a^6} \quad \textbf{(1 point)}. \end{aligned} \tag{15}$$

Equating Eqs. 7 and 8 **(1 point)**, dividing through by $\Omega - \omega \neq 0$ **(1 point)**, and solving for $\dot{a}$ gives

$$\begin{aligned} \dot{a} &= \frac{2\tau}{m\omega a} = \frac{2\tau}{m}\sqrt{\frac{a}{GM_\oplus}} \\ &= \boxed{3k_\oplus \sin(2\epsilon)\frac{m}{M_\oplus}\sqrt{\frac{GM_\oplus}{R_\oplus}}\left(\frac{R_\oplus}{a}\right)^{11/2}} \textbf{ (2 points)}. \end{aligned} \tag{16}$$

(g) Rearranging Eq. 16, **(3 points)**

$$\begin{aligned} \epsilon &= \frac{1}{2}\sin^{-1}\left(\frac{M_\oplus}{3k_\oplus m}\left(\frac{a_☾}{R_\oplus}\right)^{11/2}\sqrt{\frac{R_\oplus}{GM_\oplus}}\dot{a}_0\right) \\ &= \frac{1}{2}\sin^{-1}\left(\frac{5.976\times10^{24}}{3(0.3)(7.346\times10^{22})}\left(\frac{3.84399\times10^{8}}{6.371\times10^{6}}\right)^{11/2}\frac{1\,\mathrm{yr}}{365.2425\cdot24\cdot3600\,\mathrm{s}}\cdot\right. \\ &\qquad \left.\cdot\sqrt{\frac{6.371\times10^{6}\,\mathrm{m}}{(6.674\times10^{-11}\,\mathrm{m^3/kg/s^2})(5.976\times10^{24}\,\mathrm{kg})}}\left(38\times10^{-3}\,\mathrm{m/yr}\right)\right) \\ &= 2.4507^\circ, \end{aligned} \tag{17}$$

so $\boxed{\epsilon = 2.5^\circ}$ **(3 points)**. Accept any answer that rounds to 2.4° or 2.5°.

(h) To simplify the calculations some, we can rewrite Eq. 16 under the assumption that $k_\oplus$ and ϵ are constants as

$$\dot{a} = \dot{a}_0\left(\frac{a_☾}{a}\right)^{11/2}. \tag{18}$$

Separating **(1 point)** and integrating **(2 points)** from $t = 0$ (with $a(t = 0) \approx 0$, which is a fine approximation because the semi-major axis is raised to quite a large power) to the present (with $a = a_☾$),

$$a_☾^{-11/2}\int_0^{a_☾} a^{11/2}\mathrm{d}a = \frac{2a_☾}{13}\left(a/a_☾\right)^{13/2}\Big|_0^{a_☾} = \dot{a}_0\int_0^t \mathrm{d}t', \tag{19}$$

so

$$t = \frac{2a_☾}{13\dot{a}_0} = \frac{2(3.84399\times10^{8}\,\mathrm{m})}{13(38\times10^{-3}\,\mathrm{m/yr})} = 1.556\times10^{9}\,\mathrm{yr} \to \boxed{1.6\,\mathrm{Gyr}}. \tag{20}$$

(1 point for a correct expression, 2 points for correct answer - subtract 1 if not in terms of years)

This is smaller than the age of the Moon by almost a factor of 3! **(1 point for knowing the age of the Moon)** This suggests that ϵ was $\boxed{\text{smaller}}$ in the past (i.e., less dissipation; slower recession) **(2 points for answer and justification)**.

Gerstenkorn in the 1950s was the first to realize that backwards extrapolation of lunar recession leads to an event in which the Moon came very close to the Earth much more recently than the formation of the solar system, for which there is no geological evidence. For an example paper that takes into account many additional factors for orbital and tidal evolution (including time-dependence of ϵ), see Daher et al. (2021).

4. **(60 points)** *Stellar Odyssey*

You would like to travel to potentially habitable exoplanet Kepler 22b, located 640 light-years away from

Earth. Thankfully, your spaceship is powered by "sufficiently advanced technology" and can accelerate at 1 g (9.81 m/s^2) indefinitely. You plan to accelerate at 1 g until the halfway point of the trip, then turn around and decelerate at 1 g the rest of the way. Assume any relative velocity of Earth and Kepler 22b is negligible, and ignore gravity.

First, let's approach this problem with classical mechanics.

(a) **(5 points)** According to classical mechanics, how long will the trip take and what is the maximum velocity you will reach during the trip?

Based on these answers, is classical mechanics appropriate for describing this situation?

Now let's tackle the problem using special relativity.

Special Relativity Refresher:

A *Lorentz boost* is an origin-preserving coordinate transformation between reference frames moving with relative velocity v. To boost to a frame moving with velocity v along the $+\hat{x}$ direction, we transform space and time coordinates as follows:

$$\begin{aligned} ct' &= \gamma\,(ct - \beta x) \\ x' &= \gamma\,(x - \beta ct) \\ y' &= y \\ z' &= z \end{aligned}$$

where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ is the usual Lorentz factor.

Unlike in classical mechanics, the value of a particle's acceleration depends on the reference frame. The *proper acceleration* of a particle is defined to be the acceleration that it experiences in its own (instantaneous) rest frame.

Some identities that might be helpful in this problem:

$$\frac{d}{dx}\arctan(x) = \frac{1}{x^2+1}$$

$$\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$$

$$\int \frac{1}{\sqrt{1+x^2}}dx = \sinh^{-1}(x) + C$$

(b) **(14 points)** First, find an expression for the spacetime trajectory (or *worldline*) $x(t)$ for a massive object undergoing constant proper acceleration a, with initial condition $x(0) = v(0) = 0$. For simplicity, consider one-dimensional motion along the $\hat{x}$ direction only, and assume the object is a point mass.

Hint: You may be tempted to derive an expression relating acceleration to proper acceleration, find the acceleration in the Earth frame as a function of the ship's velocity, and integrate the resulting differential equation. This method will give the right answer, but there are more elegant approaches that involve much less calculus!

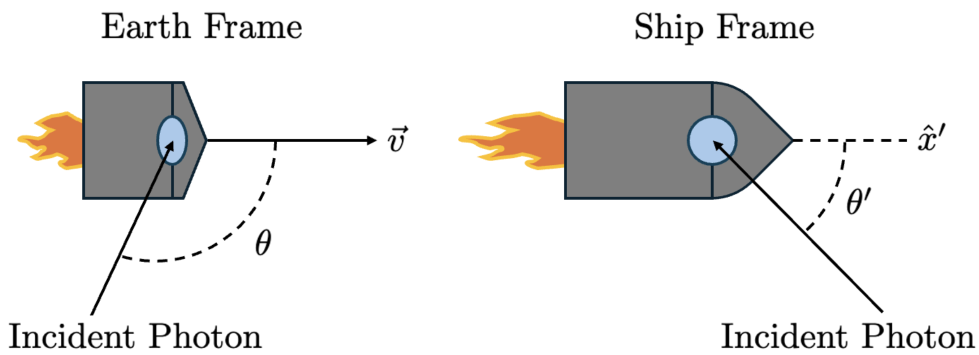
(c) **(4 points)** How long does the trip take in the Earth's reference frame? Verify that this is longer than the light travel time of 640 years.

(d) **(4 points)** What is the maximum velocity you will reach during the trip (in the Earth's reference frame)? Since this is very close to c, alternatively you can give the maximum value of γ rather than stating v.

(e) **(6 points)** How long will the trip take (proper time) from the ship's perspective?

At various times during the trip, you plan to admire the view out the ship's windows. Let's try to understand what you should expect to see when moving at highly relativistic velocities.

(f) **(10 points)**



Consider a photon hitting the ship with wavelength λ and approaching from angle θ relative to the ship's direction of travel $+\hat{x}$ (λ and θ are defined in Earth's reference frame.) $\theta = 0$ is defined to be hitting the ship from the the $+\hat{x}$ direction (therefore the photon is traveling in the $-\hat{x}$ direction.) If the ship is moving with velocity $v = \beta c$, what are the photon's observed wavelength λ' and angle θ' in the ship's reference frame?

Write your answer in terms of λ, θ, and β (and/or γ.)

(g) **(14 points)**

As you are watching out the side window, you see a planet pass by. You naively expect it to appear squished along the $\hat{x}$ direction due to relativistic length contraction. What shape does the planet actually appear to be? Prove your answer **quantitatively** using the result of (f).

Hint: A useful intermediate step is to show that

$$\frac{\sin\theta'}{\sin\theta} = \frac{d\theta'}{d\theta}$$

(h) **(3 points)**

Based on the result of (f), describe in words what you would see when looking out the front-facing window at the halfway point of the trip. Consider how different sources of light will be affected by the Lorentz boost.

Do you think looking out the front-facing window is a good idea?

Solution:

(a) In classical mechanics, we have $x = \frac{1}{2}at^2$ for uniform acceleration a starting from rest. If the total time of the trip is T, then we have $x = 320$ light years at $t = T/2$ (the halfway point.) Therefore:

$$320 \text{ light years} = \frac{1}{2}(9.81\ m/s^2)(T/2)^2$$

$$T = \boxed{49.8 \text{ years}} \ (\textbf{2 points})$$

For the first half of the trip, your velocity is $v = at$. The maximum velocity reached at $T/2$ is $v_{\max} = \frac{1}{2}(9.81\ m/s^2)(49.8 \text{ years})$

$$= \boxed{7.7 \times 10^9 \text{ m/s}}\ (\textbf{2 points})$$

This is about 26 times the speed of light, so clearly we need relativity to properly analyze this situation (**1 point**).

(b) There are many methods to solve this problem, which you can find online if you're curious. Any valid argument gets **9 points**. Partial credit breakdown is not provided due to the open-ended nature of the problem.

One particularly elegant approach is to use the *rapidity* defined as $w = \tanh^{-1}(\beta)$, which makes the derivation very simple. Another approach uses the velocity addition formula. Here I will go through a solution that doesn't rely on any external formula knowledge.

For any point along such an object's trajectory, the worldline of the object is always the same in its own instantaneous rest frame. This is because it is always the worldline of an object undergoing constant proper acceleration a, and always has initial condition $x(0) = v(0) = 0$. Therefore, whatever the worldline $x(t)$ looks like, it must be *invariant* under Lorentz boosts that transform between the ship's instantaneous reference frames at different times (up to an overall shift of origin).

Therefore, we can try to construct a worldline with this property, and see if it is the one we are looking for.

In special relativity, the invariant interval between two points is $\Delta s^2 = c^2 \Delta t^2 - \Delta x^2$ (or with the signs flipped, depending on your favored convention).

If one of these points is the origin, we can just write $\Delta s^2 = c^2t^2 - x^2$. Since the origin is preserved under Lorentz boosts, we can think of this as being an invariant quantity that's just associated with a single point in spacetime. (This is not invariant under the full set of Lorentz transformations that also includes spacetime translations, but for the sake of this argument we are only thinking about boosts.)

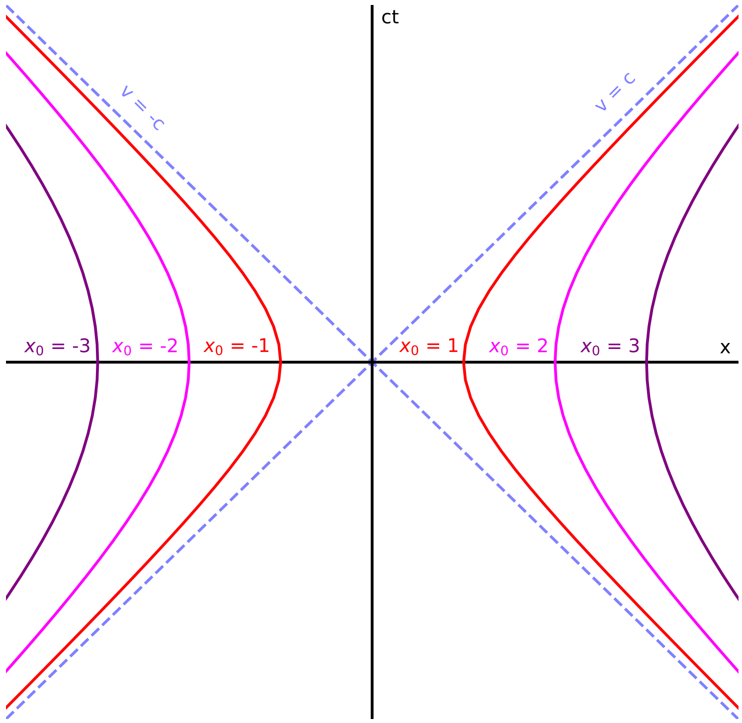
The set of all points with equal Δs^2 from the origin should be preserved under a Lorentz boost, since any point in the set will be mapped to another point in the set. Let's choose a value x_0 such that $\Delta s^2 = -x_0^2$. Then we can write

$$-x_0^2 = c^2t^2 - x^2$$

$$x(t) = \pm\sqrt{c^2t^2 + x_0^2}$$

This gives a family of worldlines parametrized by the value of x_0. Technically, each value of x_0 gives two separate worldlines, so let's take positive values of x_0 to correspond to the positive solution and negative values to correspond to the negative solution, so that $x(0) = x_0$.

Drawing these worldlines on a spacetime diagram, we can see that they are hyperbolas, and an object following them has $v = 0$ at $t = 0$.



This is indeed the correct answer; these worldlines do correspond to a particle moving at constant proper acceleration. In fact, we have constructed all the timelike worldlines with this Lorentz-boost invariance property.

Lastly, we need to actually find which worldline corresponds to the specific proper acceleration a that we want.

Using the expression for the positive solution, the acceleration at $t = 0$ for the worldline given by x_0 is

$$a = \left.\frac{d^2x(t)}{dt^2}\right|_{t=0} = \frac{c^2}{x_0}$$

Therefore we can just plug in $x_0 = \frac{c^2}{a}$:

$$x(t) = \frac{c^2}{a}\sqrt{\left(\frac{at}{c}\right)^2 + 1}$$

Note here that $x(0) = x_0$. If we want the spaceship to start at $x(0) = 0$ we simply shift the origin by $-x_0 = -\frac{c^2}{a}$:

$$\boxed{x(t) = \frac{c^2}{a}\left(\sqrt{\left(\frac{at}{c}\right)^2 + 1} - 1\right)}$$

This is the correct final answer.

- **5 points** for a fully correct final answer
- **-1 point** for a mostly correct answer that does not satisfy $x(0) = 0$
- **-2 points** for a mostly correct answer with the wrong value of the acceleration
- **No points** for any worldline that is not a hyperbola.

(c) Again we call the total length of the trip T, and we have $x = 320$ light years at $t = T/2$. Now solving for T using the expression derived in the previous part for $x(t)$ gives

$$320 \text{ light years} = \frac{c^2}{a}\left(\sqrt{\left(\frac{aT}{2c}\right)^2 + 1} - 1\right)$$

$$T = 642 \text{ years } (\textbf{3 points})$$

which is (slightly) longer than the light travel time (**1 point**).

(d) We can find the velocity $v(t)$ by differentiating the expression for $x(t)$:

$$v(t) = \frac{at}{\sqrt{1 + \left(\frac{at}{c}\right)^2}}$$

Now we can just plug in $T/2 = 321$ years to the expression for $v(t)$. This gives

$$v_{\max} = 2.997911 \times 10^8 \text{ m/s} = 0.9999954c$$

Calculators with finite numerical precision (or naively rounding to three significant figures) might struggle to tell this apart from c, so we can instead write rewrite the expression for $v(t)$ in terms of the Lorentz factor γ:

$$\gamma(t) = \frac{1}{\sqrt{1 - \frac{v(t)^2}{c^2}}} = \sqrt{1 + \left(\frac{at}{c}\right)^2}$$

$$\gamma_{\max} = \gamma(321 \text{ years}) = 331$$

- **4 points** for either the correct value of γ or v
- **-2 points** for incorrectly rounding v and not giving γ.

(e) The rate at which proper time passes on the ship is related to the rate of time passing on Earth by a factor of γ:

$$d\tau = \frac{dt}{\gamma(t)}$$

The proper time τ experienced by the ship can therefore be found by taking the integral of $\frac{1}{\gamma(t)}$ over the duration of the trip:

$$\tau = \int_0^T \frac{dt}{\gamma(t)}$$

We can use the expression for $\gamma(t)$ that we found in the previous part. The total time is twice the time taken for the first half of the trip:

$$\tau = 2\int_0^{T/2} \frac{dt}{\gamma(t)} = 2\int_0^{T/2} \frac{1}{\sqrt{1+\left(\frac{at}{c}\right)^2}}dt$$

(**3 points** for writing down this integral.) With one more substitution ($u = at/c$):

$$\tau = \frac{2c}{a}\int_0^{aT/2c} \frac{1}{\sqrt{1+u^2}}du = \frac{2c}{a}\left(\sinh^{-1}\left(\frac{aT}{2c}\right) - \sinh^{-1}(0)\right)$$

$$\tau = 12.6 \text{ years } (\textbf{3 points})$$

Surprisingly, even though relativity imposes a speed limit, the trip actually goes by much *faster* from your perspective than classical mechanics would suggest! (As long as you're not worried about how much time has passed back on Earth, that is.) You might actually live to see your destination!

From the ship's perspective, we can think of this relatively short travel time as the result of length contraction bringing the Earth and Kepler 22b much closer together.

(f) Without loss of generality, we consider the photon's direction of propagation to be in the xy plane, with $0 \leqslant \theta \leqslant \pi$. Since the system has axial symmetry about the $\hat{x}$ axis, the result we obtain should be valid for any photon trajectory in 3D.

If the photon has energy E, its four-momentum in the Earth frame is:

$$p = \frac{E}{c}\left(1, -\cos\theta, -\sin\theta, 0\right) \;\; (\textbf{2 points})$$

Using a Lorentz boost given by the ship's velocity $\beta c\hat{x}$, we can write the transformation of this four-vector into the ship's reference frame:

$$p' = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} p = \frac{E}{c}\left(\gamma + \beta\gamma\cos\theta, -\beta\gamma - \gamma\cos\theta, -\sin\theta, 0\right) \;\; (\textbf{2 points})$$

We can read off from p'^0 that this photon has energy $E' = E\gamma(1+\beta\cos\theta)$. Since $\lambda \propto \frac{1}{E}$, we have

$$\boxed{\lambda' = \frac{\lambda}{\gamma(1+\beta\cos\theta)}} \;\; (\textbf{3 points})$$

We also know that $\tan\theta' = \frac{p'^2}{p'^1}$, so we find that

$$\boxed{\theta' = \arctan\left(\frac{\sin\theta}{\gamma(\beta+\cos\theta)}\right)} \;\; (\textbf{3 points})$$

If you aren't comfortable working with four-vectors, this problem can also be solved equivalently without them, by thinking about how the wavefronts of a plane wave transform under the Lorentz boost. This solution also receives full credit and I will detail it here.

We describe the photon as a classical plane electromagnetic wave, with electric field amplitude $E(\vec{x},t) = E_0 \cos(\vec{k}\cdot\vec{x} - \omega t)$. The wavevector $\vec{k} = \frac{2\pi}{\lambda}(-\cos\theta, -\sin\theta, 0)$ gives the direction of propagation of the photon, and the angular frequency is $\omega = \frac{2\pi c}{\lambda}$. Therefore we can write this out more fully as

$$E(\vec{x},t) = E_0 \cos\left(\frac{2\pi}{\lambda}(-x\cos\theta - y\sin\theta - ct)\right)$$

Now we can transform this expression using a Lorentz boost. Note that a proper treatment would include the mixing of the electric and magnetic fields under a Lorentz boost, but this does not matter if all we care about is the spacing of the peaks and troughs of the wave.

$$E(\vec{x'},t') = E_0 \cos\left(\frac{2\pi}{\lambda}(-x\cos\theta - y\sin\theta - ct)\right)$$

$$= E_0 \cos\left(\frac{2\pi}{\lambda}(-\gamma(x' + \beta ct')\cos\theta - y'\sin\theta - \gamma(ct' + \beta x'))\right)$$

$$= E_0 \cos\left(\frac{2\pi}{\lambda}(-\gamma(\beta + \cos\theta)x' - y'\sin\theta - \gamma(1 + \beta\cos\theta)ct')\right)$$

We would like to write this in the form $E(\vec{x'},t') = E_0\cos(\vec{k'}\cdot\vec{x'} - \omega' t')$. Directly comparing, we can see $\omega' = \frac{2\pi}{\lambda}\gamma(1+\beta\cos\theta) = \omega\gamma(1+\beta\cos\theta)$. Since $\omega \propto \frac{1}{\lambda}$, we have

$$\lambda' = \frac{\lambda}{\gamma(1+\beta\cos\theta)}$$

which is in agreement with the result obtained by the four-vector method.

We can also see that $\vec{k'} = -\frac{2\pi}{\lambda}(\hat{x}'\gamma(\beta+\cos\theta) + \hat{y}'\sin\theta)$. Thus we have

$$\theta' = \arctan\frac{k'_{y'}}{k'_{x'}} = \arctan\left(\frac{\sin\theta}{\gamma(\beta+\cos\theta)}\right)$$

again in agreement with the result obtained by the four-vector method.

(g) Consider an observer at the same location as the ship at some time t, but at rest relative to the planet. At time t, some photons from the planet are arriving at the observer's/ship's location. The set of arrival angles of these photons form the image of the planet as seen by an observer. These photons also arrive at the same time in the ship's frame, since two observers always agree on the simultaneity of events at the same location and time. Therefore, we can simply take the image of the planet seen by the observer and transform it using the coordinate transformation derived in (f) to find the image seen by the ship.

For clarity, let's adopt a spherical coordinate system where $\hat{x}$ points "north", akin to the celestial sphere on Earth. We will take the approach of finding the apparent angular size of the planet along both the "declination"/"east-west" and "right ascension"/"north-south" directions, and showing that these are the same, which means that the planet appears round rather than squished.

The planet has radius r and is a distance d from the ship at time t, so that, to this observer, it subtends an angular size of $\theta_p = \frac{r}{d}$. Also, as in (f), let's say that the vector pointing from the observer to the planet makes an angle θ from the ship's direction of travel ($\hat{x}$). (In our coordinate system, the "declination" would be $90^\circ - \theta$.) Then photons from the northern and southern edges of the planet can reach the observer at angles from $\theta_- = \theta - \theta_p$ to $\theta_+ = \theta + \theta_p$.

In the ship's frame, these angles θ_- and θ_+ are transformed according to the result of (f). Let's use a first-order Taylor expansion of the result of (f), similar to a small-angle approximation. We have

$$\theta'_\pm = \theta' \pm \theta_p \frac{d\theta'}{d\theta}$$

The derivative here is somewhat annoying to evaluate, but it eventually simplifies nicely to

$$\frac{d\theta'}{d\theta} = \frac{1}{\gamma(1 + \beta\cos\theta)}$$

Plugging this in, we have

$$\theta'_\pm = \theta' \pm \frac{\theta_p}{\gamma(1 + \beta\cos\theta)}$$

The apparent angular size of the planet in the "declination" or "north-south" direction is therefore

$$\theta'_{p,\delta} = \theta'_+ - \theta'_- = \frac{2\theta_p}{\gamma(1 + \beta\cos\theta)} \text{ (\textbf{5 points})}$$

How about the "east-west" or "right ascension" direction? Naively, you might think that the apparent angular size of the planet in this direction is unchanged, since there is no length contraction along the $\hat{y}$ and $\hat{z}$ directions. However, this is not true. Let's consider the transformation of the coordinate system more carefully.

The observer in the planet's frame sees the eastern and western edges of the planet at declination $\delta = 90^\circ - \theta$. Let's say that the center of the planet is at right ascension $\alpha = 0$. Since their angular separation is $2\alpha\cos\delta = 2\theta_p$, their right ascension is $\alpha = \pm\frac{\theta_p}{\cos\delta} = \pm\frac{\theta_p}{\sin\theta}$. When we apply the Lorentz transform and shift into the ship's frame, the declination of these points changes but the right ascension does not. In the ship frame, the new angular separation is now

$$\theta'_{p,\alpha} = 2\alpha\cos\delta' = 2\alpha\sin\theta' = 2\theta_p\frac{\sin\theta'}{\sin\theta}$$

Plugging in once again the expression for θ' from (f):

$$\theta'_{p,\alpha} = 2\theta_p \frac{\sin\arctan\left(\frac{\sin\theta}{\gamma(\beta+\cos\theta)}\right)}{\sin\theta}$$

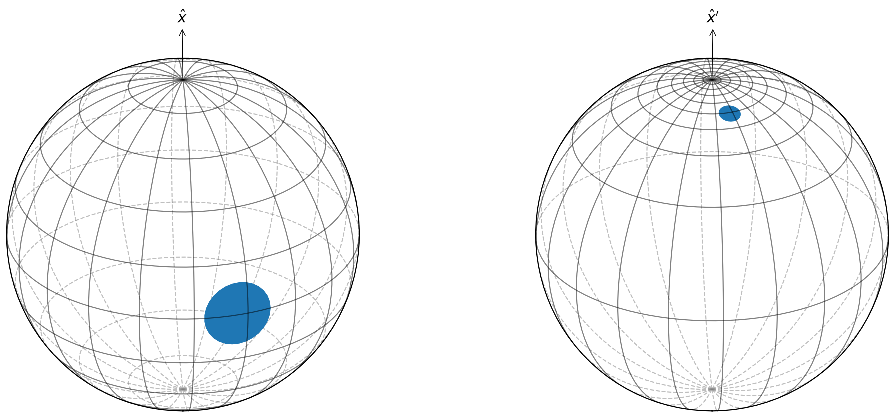
Just one more round of algebra (worked through here in full because it is not obvious that this simplifies):

$$\begin{aligned}\theta'_{p,\alpha} &= 2\theta_p \frac{\frac{\sin\theta}{\gamma(\beta+\cos\theta)}}{\sin\theta\sqrt{1 + \left(\frac{\sin\theta}{\gamma(\beta+\cos\theta)}\right)^2}} \\ &= 2\theta_p \frac{1}{\gamma(\beta+\cos\theta)\sqrt{1 + \left(\frac{\sin\theta}{\gamma(\beta+\cos\theta)}\right)^2}}\end{aligned}$$

$$= 2\theta_p \frac{1}{\sqrt{\gamma^2(\beta + \cos\theta)^2 + 1 - \cos^2\theta}}$$

$$= 2\theta_p \frac{1}{\gamma\sqrt{1 + 2\beta\cos\theta + \beta^2\cos^2\theta}} = \frac{2\theta_p}{\gamma(1 + \beta\cos\theta)} \textbf{ (5 points)}$$

Therefore, $\theta'_{p,\alpha} = \theta'_{p,\delta}$. So the planet appears to be circular, even though it is length contracted (**4 points**). This is known as the Terrell-Penrose effect. An additional aspect of this effect, which we did not derive here, is that the planet also appears to be *rotated* relative to the orientation you would see if you were at rest with respect to it.



Planet frame

Ship frame (v=0.95c)

This figure shows how the apparent image of the planet, as well as an equally-spaced spherical coordinate grid, are transformed under a Lorentz boost. For circular objects to remain circular under a coordinate transformation, the transformation must be locally angle-preserving, or conformal. This has very deep connections to group theory: the restricted Lorentz group (the group of Lorentz boosts and rotations in three dimensions) is isomorphic to the group of conformal transformations on a sphere, known as Möbius transformations. In fact, the coordinate transformation we just derived is a Möbius transformation.

(h) In the ultrarelativistic limit ($\beta \to 1, \gamma \to \infty$) the expressions from (f) give $\lambda' \to 0$ and $\theta' \to 0$ for *any* $\theta \neq \pi$.

Qualitatively, from an observer moving at ultrarelativistic speeds, this means that all the light incident on the observer will appear to contract down to a small point directly in front of the observer (around $\theta' = 0$) and be blueshifted to very high energies (of order $E' = E\gamma$). Every other direction $\theta' > 0$ will be completely dark since no photons are arriving from those directions. This effect is known as "relativistic beaming," and can be seen in the figure from part (g) by the way that the spherical coordinate grid, which is equally spaced in the Earth frame, begins to "bunch up" near the north pole in the ship frame.

The Cosmic Microwave Background emits light at a peak wavelength $\lambda_{\text{CMB}} = \frac{b}{2.73\text{K}} = 1.06\text{mm}$. With a peak γ of 331, this is blueshifted all the way to a wavelength $\lambda'_{\text{CMB}} \approx \frac{\lambda_{\text{CMB}}}{\gamma} = 3.2\mu\text{m}$. This is still outside the visible spectrum, but it turns out that the tail of the blackbody radiation curve would overlap the visible spectrum enough that the CMB would be visible to the naked eye as a faint red glow!

If it weren't for the stars, that is... A Sun-like star, emitting light at a peak wavelength $\lambda_\odot = \frac{b}{T_\odot} = 502$nm would be blueshifted all the way to a wavelength $\lambda'_\odot = 1.5$nm! This is ionizing radiation well into the range of X-rays. Perhaps you should have installed radiation shielding instead of a window.

You do not have to mention these exact things; any well reasoned and thought out answer along these lines gets full credit (**3 points**).

5. **(120 points)** *His Dark Materials*

The Cosmic Microwave Background (CMB), first discovered in 1964, is a near-uniform microwave signal from outer space, and it is now widely accepted as being the thermal blackbody radiation from the early Universe.

One of the greatest joint predictions of the Standard Model of particle physics and of modern cosmology is the existence of an analogous *Cosmic Neutrino Background*, or CNB. While the CNB has never been directly observed, its existence is nearly unambiguous in light of our modern understanding of physics. One of the most shocking things about the CNB is that, if it exists, it is possible to derive the ratio of its energy density $\rho_\nu c^2$ to the energy density $\rho_\gamma c^2$ of the CMB solely through a combination of particle physics, thermodynamics, and cosmology. Consequently, using modern CMB measurements, one can calculate the energy density of the CNB. In this problem, you will do exactly this.

(Note: This problem consists of two parts. Part A explores the underlying particle physics and thermodynamics and is worth more points, while Part B applies these results to cosmology and is shorter. In particular, these parts are ordered by content, not difficulty, and can be attempted in either order (they are independent). We strongly encourage you to try as much as you can.)

Part A: Particle in a Box (90 points)

We will begin by establishing the key underlying thermal physics for how the energy of a system of particles depends on their temperature. Below, we list observed particle properties from Earth-based particle physics experiments (and their corresponding decoupling or annihilation temperatures according to our cosmological and quantum models).

Particle	**Symbol**	**Mass**	g_{anti}	g_{spin}	N_{flavor}	**Type**	**Decouple/Annihilate Temp**
Electron	e^+/e^-	m_e	2	2	1	Fermion	$\approx 5 \times 10^9$ K
Photon	γ	0	1	2	1	Boson	≈ 3000 K
Neutrino	ν	≈ 0	2	1	3	Fermion	$\approx 10^{10}$ K
All Others	–	–	–	–	–	–	$> 10^{12}$ K

Note that while experiments such as neutrino oscillations place a nonzero lower bound on the neutrino mass, we assume it is negligibly small, so treat neutrinos as massless throughout the problem. The antiparticle of the electron is the positron, and the two have the same mass. You may also find the integrals

$$\int_0^\infty \frac{x^n}{e^x - 1} dx = n!\ \zeta(n+1) \text{ and } \int_0^\infty \frac{x^n}{e^x + 1} dx = n!\ (1 - \frac{1}{2^n})\ \zeta(n+1)$$

helpful, where $\zeta(2) = \pi^2/6$, $\zeta(3) \approx 1.202$, and $\zeta(4) = \pi^4/90$. $n!$, or n factorial, denotes the product of the integers from 1 to n, inclusive. The identities

$$|r| < 1 \to \sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \text{ and } \sum_{k=0}^{\infty} k r^k = \frac{r}{(1-r)^2}$$

may also prove useful.

In the Standard Model of particle physics, energy modes associated with each particle type have a certain number of degrees of freedom g, which may arise due to a particle being distinct from its antiparticle (so positrons count as additional "degrees of freedom" for electrons), it having more than one observed spin state, or it arising in several flavors. Thus, to calculate the total energy stored in all modes for a given particle, it suffices to calculate the energy in one mode and multiply by the number of modes g.

(a) **(2 points)** Based on the above table, propose and justify a chronological ordering of the four events below in the history of the Universe. You must give valid reasoning for credit.

  (i) The overwhelming majority of electrons and positrons pairwise annihilate each other.

  (ii) Photons decouple from ordinary matter and form the Cosmic Microwave Background.

  (iii) Neutrinos decouple from ordinary matter and form the Cosmic Neutrino Background.

  (iv) All other particles either decouple or annihilate.

(b) **(2 points)** For each (named) particle in the above table, calculate g by multiplying the three contributions to g in the above table (including N_{flavor}). Comment on any practical slight deviations to any of the N values, give an estimate, and briefly explain why.

(c) **(1 point)** Later in this problem, we will derive that each fermionic degree of freedom contributes $\frac{7}{8}$ of the energy density as each bosonic degree of freedom; adding this multiplicative correction gives us a new count $g^\star$. At sufficiently early times (just after the Big Bang), none of the Standard Model particles, such as quarks, had yet decoupled or annihilated, making all degrees of freedom active. Estimate the total effective number of degrees of freedom $g^\star_{tot}$ at this point due to all Standard Model particles.

Throughout the rest of this problem, use per-particle g and not $g^\star$.

We will now begin by exploring the basic underlying principles from thermodynamics and statistical mechanics that we will use in this problem. In statistical mechanics, an open, macroscopic system that is in both thermal (exchange of heat) and chemical (exchange of particles) equilibrium with its environment is known as a **grand canonical ensemble**. For now, we only consider identical particles of a given particle type. In such a system, the chemical potential μ denotes the increase in internal energy associated with adding one extra particle to the **environment** (not the system). In other words, transferring a particle from the environment to a state of energy ε in the system has a corresponding total energy change $\varepsilon - \mu$. For a system at temperature T, we write $\beta = 1/(k_B T)$ for convenience. Together, the parameters β and μ characterize the behavior of our system.

First, consider a simple system where particles in the system can only be in one state of energy ε. Relative to a system with zero particles in the system, a state with n particles has energy $E_n = n\varepsilon$. According to statistical mechanics, the probability of a system being in a state with energy E is proportional to $e^{-\beta(E-n\mu)}$. We define the **grand partition function** Ξ as the sum of $e^{-\beta(E-n\mu)}$ over all states, so here we have,

$$\Xi = \sum_n e^{-\beta n(\varepsilon-\mu)},$$

where n runs over all possible numbers of particles in this system. In general, we can use this to express the probability of a system being in a state with energy E and n particles as $e^{-\beta(E-n\mu)}/\Xi$.

(d) **(5 points)** Fermions are particles with half-integer spin. According to the Pauli exclusion principle, no two identical fermions can occupy the same quantum state, so $n \in \{0, 1\}$ above.

  (i) Write out the expression for Ξ for the above system if the particle in question is a fermion. Give the probabilities of the system being in the $n = 0$ and $n = 1$ states.

  (ii) Use this to calculate the expected value (average value) of the number of particles in the system $\langle n_\varepsilon \rangle$ and the expected energy of the system $\langle E_\varepsilon \rangle$.

(e) **(5 points)** Bosons are particles with integer spin and face no such restrictions. As such, $n \geqslant 0$ can be any nonnegative integer.

(i) Write out and simplify the expression for Ξ for the above system if the particle in question is a boson. Give the probability of the system being in the state with n particles in terms of n.

(ii) Use this to calculate the expected number of particles in the system $\langle n_\varepsilon \rangle$ and the expected energy of the system $\langle E_\varepsilon \rangle$.

Now consider a system with many individual possible states for particles to occupy. A given state with energy ε has the above probability distribution for its occupancy, independent of all other states. For a macroscopic system, we can simply take the overall energy and number of particles to be their expected values, giving

$$E = \sum_\varepsilon \varepsilon \langle n_\varepsilon \rangle \text{ and } N = \sum_\varepsilon \langle n_\varepsilon \rangle,$$

where the sum is over all states (indexed by their energies ε). Partition functions over independent states multiply, so

$$\Xi = \prod_\varepsilon \Xi_\varepsilon,$$

where Ξ_ε is the partition function for that state alone (independently of all others).

(f) **(5 points)** Verify the identity

$$E = -\left(\frac{\partial \ln \Xi}{\partial \beta}\right)_\mu + \mu N,$$

holds for both the fermionic and bosonic cases, where, as usual, the subscript to the bottom-right of a partial derivative denotes keeping that variable constant while taking the partial derivative.

We will now consider a macroscopic system of identical particles in a grand canonical ensemble of a known V, μ, and β. We will make the following assumptions about the system:

- Particles do not interact with each other. (Note that this does *not* mean that the states of all particles are independent, as principles like the Pauli exclusion principle may still apply. Rather, particles do not exert forces on each other and thus may be treated with the framework we establish.)
- The system is homogeneous and isotropic, so the energy of a particle ε solely depends on the magnitude of its momentum p.

In particular, we assume that there is no external potential, meaning that we have a simple relationship between energy and momentum.

(g) **(5 points)** Consider a particle of mass m (possibly equal to zero) and momentum p. Give your answers to the below questions in terms of m, p, and physical constants.

(i) According to relativity, what is the **general** expression for ε in terms of the above variables that is valid for all speeds?

(ii) Simplify your answer above if the particle is either massless or ultrarelativistic ($p \gg mc$).

(iii) Instead, assume the particle is massive and not relativistic ($p \ll mc$). Give the Taylor expansion of the formula for ε to leading nonzero order in p.

In the below subparts, we will denote the energy of a state as being a function of its momentum via $\varepsilon(p)$. Now, according to quantum statistics, the number of single particle states in a given volume d^3p of 3D momentum space (the state of all possible momenta of a particle) of a system with volume V and degeneracy g is $gV/(2\pi\hbar)^3 \cdot d^3p$. Note that integrating over all of momentum space is equivalent to integrating each dimension of momentum (p_x, p_y, and p_z) from negative infinity to infinity. *(It may be helpful that in an isotropic system, this is equivalent to integrating $4\pi^2 dp$ from zero to infinity.)*

(h) **(10 points)** Write the total energy E and number of particles N as integrals over the **magnitude** of the momentum p. Express your answers in terms of g, V, $\hbar$, β, μ, and the general functional form $\varepsilon(p)$. **You do not need to evaluate the integrals.** *(You may either do this separately for both fermions and bosons or do them together with a cleverly placed $\pm$ sign. If you do the latter, specify which branch corresponds to what particle type.)*

Now, in the early Universe, photons could be created and destroyed freely in various events, so their chemical potential was $\mu_\gamma = 0$.

(i) **(10 points)** Calculate the energy density of a photon gas $u_\gamma = E_\gamma/V$ in terms of its temperature T and fundamental constants. State how the energy density scales with temperature.

(j) **(5 points)** Repeat this calculation for a massless or ultrarelativistic fermion gas of temperature T and degrees of freedom g, again assuming that the chemical potential is zero. Substitute the values of g for both electrons and neutrinos.

We will now turn our attention to the entropy S of the system. Letting P be the pressure of the system, define the grand potential Ω as

$$\Omega = -k_B T \ln \Xi = E - TS - \mu N = -PV.$$

Also recall the differential form of the first law of thermodynamics:

$$dE = TdS - PdV + \mu dN$$

You may freely use all the above equations without proof. Be careful to not omit (relevant) terms when working with differentials.

(k) **(5 points)** Prove the identity

$$S = -\left(\frac{\partial \Omega}{\partial T}\right)_{V,\mu},$$

where we keep the volume and chemical potential constant.

(l) **(10 points)** Using the same integral form as earlier, express S as an integral over p using V, β, μ, $\varepsilon(p)$, and fundamental constants. Be sure to handle both the fermion and boson cases.

(m) **(15 points)** Now, consider only massless or ultrarelativistic particles. Express the energy E in terms of the pressure P and the volume V. *(It may be helpful to handle fermions and bosons separately and use integration by parts on an integral expression for* $\ln \Xi$. *Alternatively, you may also choose to use a more classical thermodynamic argument to directly obtain an expression for both cases simultaneously. Either way, you* ***must*** *give a complete derivation for credit.)*

(n) **(10 points)** Use this to write the entropy density $s = S/V$ in terms of the energy density $u = E/V$, the number density $n = N/V$, μ, and T for massless or ultrarelativistic particles. Give a second simplified answer for the case $\mu = 0$.

Part B: Nothing, Except Everything (30 points)

As the Universe continued to expand, after neutrinos decoupled from ordinary matter, electrons and positrons became less relativistic, after which virtually all electrons and positrons annihilated each other. Since photons had not yet decoupled from ordinary matter, the energy from the electrons and positrons went into heating the photon gas, but it did not heat the neutrino background. As a result, while neutrinos and photons were at the same temperature before the electron annihilation, the photon gas became warmer than the neutrino background. We will now compute the corresponding temperature and energy density ratios of the present-day photon and neutrino backgrounds.

You may use the results $u \propto gg'T^4$ (energy density) and $s \propto gg'T^3$ (entropy density) for ultrarelativistic particles from the previous section throughout this section. Here, g is the number of degrees of freedom from (b) and g' is 1 if the particle is a boson or 7/8 if it is a fermion.

Assume a smooth, uniform expanding universe in equilibrium with itself. Consider a box of volume V that expands with the Universe (it is comoving), so $V \propto a^3$, where a is the scale factor. (Specifically, any distance d in the present day corresponds to two points that were separated by a distance ad in the past at a point where the scale factor was a, after which the points moved apart due to the expansion of the Universe.)

(o) **(5 points)** First of all, we will show that the total energy in the box is likely *not* conserved. To do this, consider a universe made out of only a photon gas and state the proportionality relations for how T and $U = uV$ scale with a.

(p) **(10 points)** On the other hand, under these assumptions, the total entropy in the comoving box *is* conserved. Indeed, according to the First Law of Thermodynamics,

$$dU = -PdV + TdS \to d(uV) = -PdV + TdS.$$

Use this in conjunction with the Friedmann continuity equation

$$\dot{\rho} + 3H(\rho + \frac{P}{c^2}) = 0$$

to show that $dS = 0$ over a differential timestep dt.

Before neutrino decoupling, all three relevant types of particles (electrons, neutrinos, and photons) were ultrarelativistic and were at the same temperature. After neutrino decoupling, the comoving entropies of both the neutrino system and the electron plus photon system stayed constant. In particular,

$$\frac{s_e a^3 + s_\gamma a^3}{s_\nu a^3} = \text{const.}$$

(q) **(10 points)** By evaluating the above both before electron annihilation (while electrons were still ultrarelativistic) and after electron annihilation (where there were essentially no electrons), calculate the present-day photon to neutrino temperature ratio T_γ/T_ν. Evaluate T_ν to three significant figures given $T_\gamma \approx 2.725$ K.

(r) **(5 points)** Calculate the energy density ratio ρ_ν/ρ_γ and evaluate this to three significant figures.

Solution: Solution to Question 5

(a) Notice that as the Universe expands, it cools, so all we need to do is sort in decreasing order of T. This corresponds to $\boxed{\text{(iv), (iii), (i), (ii)}}$.

(b) We multiply to get $g_e = 2 \cdot 2 \cdot 1 = \boxed{4}$, $g_\gamma = 1 \cdot 2 \cdot 1 = \boxed{2}$, and $g_\nu = 2 \cdot 1 \cdot 3 = \boxed{6}$. This is sufficient for full credit. **(Note that according to modern research, there is a slight correction to the effective neutrino N, bringing it closer to 3.044.)**

(c) Any response receives full credit.

(d) We have the following:

(i) $\Xi = \boxed{1 + e^{-\beta(\varepsilon-\mu)}}$. Consequently, the probability it is in the $n = 0$ state is $\boxed{\dfrac{1}{1 + e^{-\beta(\varepsilon-\mu)}}}$ and the probability it is in the $n = 1$ state is $\boxed{\dfrac{e^{-\beta(\varepsilon-\mu)}}{1 + e^{-\beta(\varepsilon-\mu)}}}$.

(ii) The expected number of particles in the system is just the probability that the system is in the $n = 1$ state, which is $\boxed{\dfrac{e^{-\beta(\varepsilon-\mu)}}{1 + e^{-\beta(\varepsilon-\mu)}}}$. The expected energy is just $(\varepsilon)\langle n_\varepsilon \rangle = \boxed{\varepsilon\dfrac{e^{-\beta(\varepsilon-\mu)}}{1 + e^{-\beta(\varepsilon-\mu)}}}$. Note that we can re-express the expected number of particles as $\frac{1}{e^{\beta(\varepsilon-\mu)}+1}$.

(e) (i) Now, the formula for Ξ is slightly more nontrivial, but it can be evaluated as a geometric series:

$$\Xi = \sum_{n\geqslant 0} e^{-n\beta(\varepsilon-\mu)} = \boxed{\frac{1}{1-e^{-\beta(\varepsilon-\mu)}}}.$$

This gives that the probability of being in a state with n particles is just

$$e^{-n\beta(\varepsilon-\mu)}/\Xi = \boxed{e^{-n\beta(\varepsilon-\mu)}(1-e^{-\beta(\varepsilon-\mu)})}.$$

(ii) The expected number of particles is just

$$\sum_n ne^{-n\beta(\varepsilon-\mu)}(1-e^{-\beta(\varepsilon-\mu)}) = (1-e^{-\beta(\varepsilon-\mu)})\sum_{n\geqslant 0} ne^{-n\beta(\varepsilon-\mu)} = (1-e^{-\beta(\varepsilon-\mu)})\cdot\frac{e^{-\beta(\varepsilon-\mu)}}{(1-e^{-\beta(\varepsilon-\mu)})^2}$$

thus giving

$$\langle n_\varepsilon\rangle = \boxed{\frac{1}{e^{\beta(\varepsilon-\mu)}-1}} \to \langle E_\varepsilon\rangle = \boxed{\frac{\varepsilon}{e^{\beta(\varepsilon-\mu)}-1}}$$

(f) Notice that $\ln\Xi = \sum_\varepsilon \ln\Xi_\varepsilon$, and

$$\left(\frac{\partial\ln\Xi_\varepsilon}{\partial\beta}\right)_\mu = \frac{1}{\Xi_\varepsilon}\left(\frac{\partial\Xi_\varepsilon}{\partial\beta}\right)_\mu = \frac{\sum_n -n(\varepsilon-\mu)e^{-\beta n(\varepsilon-\mu)}}{\sum_n e^{-\beta n(\varepsilon-\mu)}} = \frac{\mu\sum_n ne^{-\beta n(\varepsilon-\mu)} - \sum_n n\varepsilon e^{-\beta n(\varepsilon-\mu)}}{\sum_n e^{-\beta n(\varepsilon-\mu)}},$$

which is just $\mu\langle n_\varepsilon\rangle - \langle E_\varepsilon\rangle$ for this state. Summing over all states gives

$$\left(\frac{\partial\ln\Xi}{\partial\beta}\right)_\mu = \sum_\varepsilon\left(\frac{\partial\ln\Xi_\varepsilon}{\partial\beta}\right)_\mu = \sum_\varepsilon \mu\langle n_\varepsilon\rangle - \langle E_\varepsilon\rangle = \mu N - E,$$

as desired.

(g) We have the following results from relativity:

(i) The general form is $\varepsilon = \boxed{\sqrt{(pc)^2+(mc^2)^2}}$. This can be derived from $\varepsilon = \gamma mc^2$ and $p = \gamma mv$.

(ii) Dropping the m term gives $\varepsilon = \boxed{pc}$.

(iii) This is just

$$\varepsilon = mc^2\sqrt{1+(\frac{pc}{mc^2})^2} \approx mc^2(1+\frac{p^2}{2m^2c^2}) = \boxed{mc^2+\frac{p^2}{2m}}.$$

Observe that this is just rest mass-energy plus the classical kinetic energy.

(h) Combining our results from (d) and (e), observe the unified form

$$\langle n_\varepsilon\rangle = \frac{1}{e^{\beta(\varepsilon-\mu)}\pm 1},$$

where we have the + sign (top) for fermions and the − sign (bottom) for bosons. We will use this going forward. Now notice that integrating d^3p over $\mathbb{R}^3$ is equivalent to integrating $4\pi p^2 dp$ over $p\in[0,\infty)$. This gives the integral forms

$$N = \int_0^\infty \frac{gV}{(2\pi\hbar)^3}\cdot\frac{1}{e^{\beta(\varepsilon(p)-\mu)}\pm 1}\cdot 4\pi p^2 dp$$

and

$$E = \int_0^\infty \frac{gV}{(2\pi\hbar)^3}\cdot\frac{\varepsilon(p)}{e^{\beta(\varepsilon(p)-\mu)}\pm 1}\cdot 4\pi p^2 dp.$$

(i) We substitute $g = 2$, $\mu = 0$, $\varepsilon(p) = pc$, the $-$ sign, and later the substitution $x = \beta pc$ to get

$$u = \frac{E}{V} = \int_0^\infty \frac{2}{(2\pi\hbar)^3} \cdot \frac{pc}{e^{\beta pc} - 1} \cdot 4\pi p^2 dp = \frac{c\pi}{(\pi\hbar)^3} \int_0^\infty \frac{p^3 dp}{e^{\beta pc} - 1} = \frac{\pi}{(\pi\hbar)^3 \beta^4 c^3} \int_0^\infty \frac{x^3 dx}{e^x - 1}$$

$$= \frac{\pi}{(\pi\hbar)^3 \beta^4 c^3} \cdot 6 \cdot \frac{\pi^4}{90} = \boxed{\frac{\pi^2 k_B^4}{15 c^3 \hbar^3} T^4},$$

giving us the expected

$$\boxed{u \propto T^4}$$

from the Stefan-Boltzmann law.

(j) We retrace the above calculation with the + sign, giving:

$$u = \frac{E}{V} = \int_0^\infty \frac{g}{(2\pi\hbar)^3} \cdot \frac{pc}{e^{\beta pc} + 1} \cdot 4\pi p^2 dp = \frac{gc\pi}{2(\pi\hbar)^3} \int_0^\infty \frac{p^3 dp}{e^{\beta pc} + 1} = \frac{g\pi}{2(\pi\hbar)^3 \beta^4 c^3} \int_0^\infty \frac{x^3 dx}{e^x + 1}$$

$$= \frac{g\pi}{2(\pi\hbar)^3 \beta^4 c^3} \cdot 6 \cdot \frac{7}{8} \cdot \frac{\pi^4}{90} = \boxed{\frac{7 g \pi^2 k_B^4}{240 c^3 \hbar^3} T^4},$$

which gives $g_e = 4 \rightarrow \frac{7\pi^2 k_B^4}{60 c^3 \hbar^3} T^4$ for electrons and $g_\nu = 6 \rightarrow \frac{7\pi^2 k_B^4}{40 c^3 \hbar^3} T^4$ for neutrinos at zero chemical potential and at high temperatures.

(k) Taking the differential of the given equation,

$$d\Omega = dE - d(TS) - d(\mu N) = (TdS - PdV + \mu dN) - (TdS + SdT + \mu dN + N d\mu),$$

so

$$d\Omega = -PdV - SdT - N d\mu \rightarrow \left(\frac{\partial \Omega}{\partial T}\right)_{V,\mu} = -S,$$

as desired. As stated in the problem text, it is easy to mess up the differentials here by omitting terms: any term with a first-order differential **must** be tracked, as we have above.

(l) Notice that the given equation implies

$$TS = E - \mu N + k_B T \ln \Xi \rightarrow S = k_B \beta((E - \mu N) + \frac{1}{\beta} \ln \Xi).$$

Now using the unified form (which we later convert to an integral)

$$\ln \Xi = \sum_\varepsilon \pm \ln(1 \pm e^{-\beta(\varepsilon - \mu)})$$

with, as before, top signs denoting fermions and bottom signs denoting bosons, we have:

$$s = \frac{S}{V} = \int_0^\infty \frac{g}{2\pi^2 \hbar^3} k_B (\beta \frac{(\varepsilon(p) - \mu)}{e^{\beta(\varepsilon(p) - \mu)} \pm 1} \pm \ln(1 \pm e^{-\beta(\varepsilon(p) - \mu)})) \cdot p^2 dp.$$

(m) While it is possible to do this with statistical mechanics, we first sketch a clean, purely classical argument that extends to both the boson and the fermion case. Consider a particle moving at speed c and x-velocity v_x in an axis-aligned box of side length L. Upon reaching either of the boundaries of the container, say that the particle reflects elastically. Then the average time between collisions with a given face in the yz plane is

$$\delta t = \frac{2L}{v_x}$$

and the average momentum imparted per collision is

$$\delta p = 2p_x = \frac{2Ev_x}{c^2}$$

. Then the average force imparted is

$$F = \frac{\delta p}{\delta t} = \frac{Ev_x^2}{Lc^2},$$

giving an average pressure

$$P = \frac{F}{L^2} = \frac{Ev_x^2}{c^2L^3} = \frac{Ev_x^2}{c^2V},$$

where V is the volume. But $\langle v_x^2 \rangle = c^2/3$ by symmetry over the three dimensions, so

$$\langle P \rangle = \frac{E}{3V}.$$

From a macroscopic sum over all particles, this translates to

$$E = \boxed{3PV},$$

which is valid for both fermions and bosons.

We now describe the alternative statistical mechanics argument. For brevity, we will only do the fermion case: the boson case is analogous. Observe that we have

$$\ln \Xi = \frac{4\pi gV}{(2\pi\hbar)^3}\int_0^\infty \ln(1+e^{-\beta(pc-\mu)})p^2\,dp$$

by our general integral construction. We now use integration by parts on the integral with

$$A = \ln(1+e^{-\beta(pc-\mu)}) \to dA = \frac{-\beta ce^{-\beta(pc-\mu)}dp}{1+e^{-\beta(pc-\mu)}} = \frac{-\beta c}{e^{\beta(pc-\mu)+1}}dp$$

and

$$dB = p^2dp \to B = \frac{1}{3}p^3$$

giving

$$\int_0^\infty \ln(1+e^{-\beta(pc-\mu)})\cdot p^2dp = \left[\ln(1+e^{-\beta(pc-\mu)})\cdot\frac{1}{3}p^3\right]_0^\infty - \int_0^\infty \frac{-\beta c}{e^{\beta(pc-\mu)+1}}dp\cdot\frac{1}{3}p^3.$$

The first term vanishes at both zero and infinity (as $p\to\infty$ it approaches $\frac{1}{3}p^3e^{-\beta(pc-\mu)}$, which goes to zero because of the exponential). Furthermore, the second integral is exactly the integral for E scaled by a factor:

$$\ln \Xi = \frac{\beta}{3}\cdot\frac{4\pi gV}{(2\pi\hbar)^3}\cdot\int_0^\infty \frac{pc\cdot p^2dp}{e^{\beta(pc-\mu)}+1} = \frac{\beta}{3}\cdot E,$$

so returning to our original Ω identity gives

$$PV = -\Omega = k_BT\ln\Xi = \frac{1}{\beta}\cdot\frac{\beta E}{3} = \frac{E}{3} \to E = \boxed{3PV},$$

finishing.

(n) We simply substitute $E = 3PV \to PV = \frac{1}{3}E$ into the grand potential equation, giving

$$TS = E + PV - \mu N = \frac{4}{3}E - \mu N \to s = \boxed{\frac{4u}{3T} - \frac{\mu n}{T}}.$$

Now if $\mu = 0$, we have $s = \boxed{\frac{4u}{3T}}$, which scales as T^3. This will be useful in the next section.

(o) By cosmology and what we have earlier, $\boxed{T \propto a^{-1}}$, $V \propto a^3$, and $u \propto T^4 \propto a^{-4}$, giving $uV = \boxed{U \propto a^{-1}}$. So total energy is not an invariant here.

(p) We have

$$-PdV + TdS = d(uV) = Vdu + udV = c^2(Vd\rho + \rho dV) \to TdS = Vc^2 d\rho + \rho c^2 dV + PdV.$$

Dividing the right hand side by Vdt and noticing that $\frac{dV}{V} = 3\frac{da}{a} = 3H$,

$$\frac{TdS}{Vdt} = \dot{\rho}c^2 + 3H(\rho c^2 + P) = 0,$$

as desired.

(q) We substitute $s \propto gg'T^3$, so canceling the a^3 terms,

$$\frac{4 \cdot \frac{7}{8} \cdot T_e^3 + 2 \cdot 1 \cdot T_\gamma^3}{6 \cdot \frac{7}{8} \cdot T_\nu^3} = C$$

for some constant C. At sufficiently high Universe temperatures, all three species were in thermal equilibrium, giving $C = \frac{\frac{7}{2}+2}{\frac{21}{4}} = \frac{22}{21}$. At sufficiently low universe temperatures, there were no electrons (so functionally $T_e = 0$), giving,

$$\frac{2 \cdot T_\gamma^3}{\frac{21}{4} \cdot T_\nu^3} = \frac{22}{21} \to \frac{T_\gamma}{T_\nu} = \boxed{(\frac{11}{4})^{1/3}}.$$

Numerically, this gives $T_\nu \approx \boxed{1.95 \text{ K}}$.

(r) We have

$$\rho_\nu/\rho_\gamma = \frac{g_\nu g'_\nu T_\nu^4}{g_\gamma g'_\gamma T_\gamma^4} = 3 \cdot \frac{7}{8} \cdot (\frac{4}{11})^{4/3} \approx \boxed{0.681}.$$

Scoring: give full error carried forward. Penalize each individual mistake at either 1 or 2 points ("proportional" to the problem point value). Give partial credit proportional to progress into the problem.