USAAAO 2026 - First Round

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1. A planet moves in a circular orbit in a fixed plane with angular velocity ω, and let $\hat{n}$ be a unit vector perpendicular to the orbital plane. Let the planet's velocity vector as a function of time be given by

$$\vec{v}(t) = v_0\big(\cos(\omega t)\,\hat{i} + \sin(\omega t)\,\hat{j}\big)$$

Define $\vec{a}(t) = \frac{d\vec{v}}{dt}$. Which of the following statements is **correct**? *(Here, $\hat{i}$ and $\hat{j}$ are orthogonal unit vectors in the orbital plane.)*

(a) $\vec{a}$ is always parallel to $\vec{v}$

(b) $\vec{a}$ has a magnitude $v_0\omega$ and is perpendicular to $\vec{v}$

(c) $\vec{a}$ is zero since the magnitude of $\vec{v}$ is constant

(d) $\vec{v} \cdot \vec{a} = v_0^2\omega$

(e) $\vec{a}$ is parallel to $\hat{n}$

Solution: Given

$$\vec{v}(t) = v_0\big(\cos(\omega t)\,\hat{i} + \sin(\omega t)\,\hat{j}\big),$$

which represents a velocity vector of constant magnitude v_0 rotating uniformly in the orbital plane.

Time derivative

The acceleration is defined as

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = v_0\big(-\omega\sin(\omega t)\,\hat{i} + \omega\cos(\omega t)\,\hat{j}\big).$$

Magnitude of the acceleration

$$|\vec{a}| = v_0\omega\sqrt{\sin^2(\omega t) + \cos^2(\omega t)} = v_0\omega.$$

Direction of the acceleration

The dot product of the velocity and acceleration is

$$\vec{v} \cdot \vec{a} = v_0^2 \omega \Big(\cos(\omega t)(-\sin(\omega t)) + \sin(\omega t)\cos(\omega t) \Big) = 0.$$

Since the dot product vanishes, the acceleration vector is perpendicular to the velocity vector.

Equivalently, because the speed of the planet is constant,

$$\frac{d}{dt}(\vec{v} \cdot \vec{v}) = 0 \quad \Rightarrow \quad 2\vec{v} \cdot \vec{a} = 0,$$

which again implies that $\vec{a}$ is perpendicular to $\vec{v}$.

Evaluation of answer choices

- Option (a) is incorrect because the acceleration of uniform circular motion is perpendicular, not parallel, to the velocity.
- Option (b) is correct: the acceleration has magnitude $v_0 \omega$ and is perpendicular to the velocity.
- Option (c) is incorrect because a constant speed does not imply zero acceleration; the direction of the velocity is changing.
- Option (d) is incorrect because $\vec{v} \cdot \vec{a} = 0$, not $v_0^2 \omega$.
- Option (e) is incorrect because the acceleration lies in the orbital plane, whereas $\hat{n}$ is perpendicular to that plane.

Answer: B

2. A planet of mass m orbits a star of mass M in an elliptical orbit with semi-major axis a and eccentricity e where $M \gg m$. Which of the following statements is **true**?

(a) The orbital speed is maximized at aphelion due to conservation of energy

(b) The planet moves in a perfect circle around the star regardless of eccentricity

(c) The orbital period depends on eccentricity through Kepler's Third Law

(d) The Roche limit increases if the planet's density decreases

(e) All five Lagrange points correspond to stable equilibria

Solution:

(a) Orbital speed is maximized at aphelion. FALSE. Speed is maximized at perihelion due to conservation of angular momentum.

(b) The planet moves in a perfect circle around the star regardless of eccentricity. FALSE. The problem states the orbit is elliptical, so the planet does not move in a perfect circle.

(c) Orbital period depends on eccentricity. FALSE. Kepler's Third Law gives $T^2 \propto a^3$, independent of eccentricity.

(d) Roche limit increases if planet density decreases. TRUE. Roche limit scales as

$$r_R \propto \left(\frac{\rho_\star}{\rho_p}\right)^{1/3}.$$

Lower planet density ρ_p increases r_R.

(e) All Lagrange points stable. FALSE. L_1, L_2, L_3 are unstable; only L_4, L_5 may be stable.

Answer: D

3. Astronomers in the United States observe a spiral galaxy and measure its rotation curve. They find that beyond a radius r_0, the orbital speed of stars remains approximately constant at v_0. Which of the following conclusions is **most accurate**?

(a) The mass density of the galaxy must decrease faster than $1/r^2$

(b) The total enclosed mass becomes constant for $r > r_0$

(c) The luminosity profile directly traces the mass distribution

(d) The enclosed mass within radius r grows approximately linearly with r

(e) The galaxy cannot contain a central supermassive black hole

Solution: Given a spiral galaxy with constant orbital speed $v(r) = v_0$ for $r > r_0$.

Step 1: Use circular motion

$$\frac{GM(r)}{r^2} = \frac{v_0^2}{r} \quad \Rightarrow \quad M(r) = \frac{v_0^2}{G} r$$

Step 2: Physical interpretation - Enclosed mass increases linearly with radius, even where visible matter is sparse. - This is evidence of a dark matter halo.

Step 3: Evaluate choices

(a) The mass density of the galaxy must decrease faster than $1/r^2$.
This is incorrect because if the density decreased faster than $1/r^2$, the orbital speed would decrease with radius. A flat rotation curve requires $\rho(r) \sim 1/r^2$ at large radii.

(b) The total enclosed mass becomes constant for $r > r_0$.
This is incorrect because $v^2 = GM(r)/r$ and v = constant imply $M(r) \propto r$, so the enclosed mass continues to increase with radius.

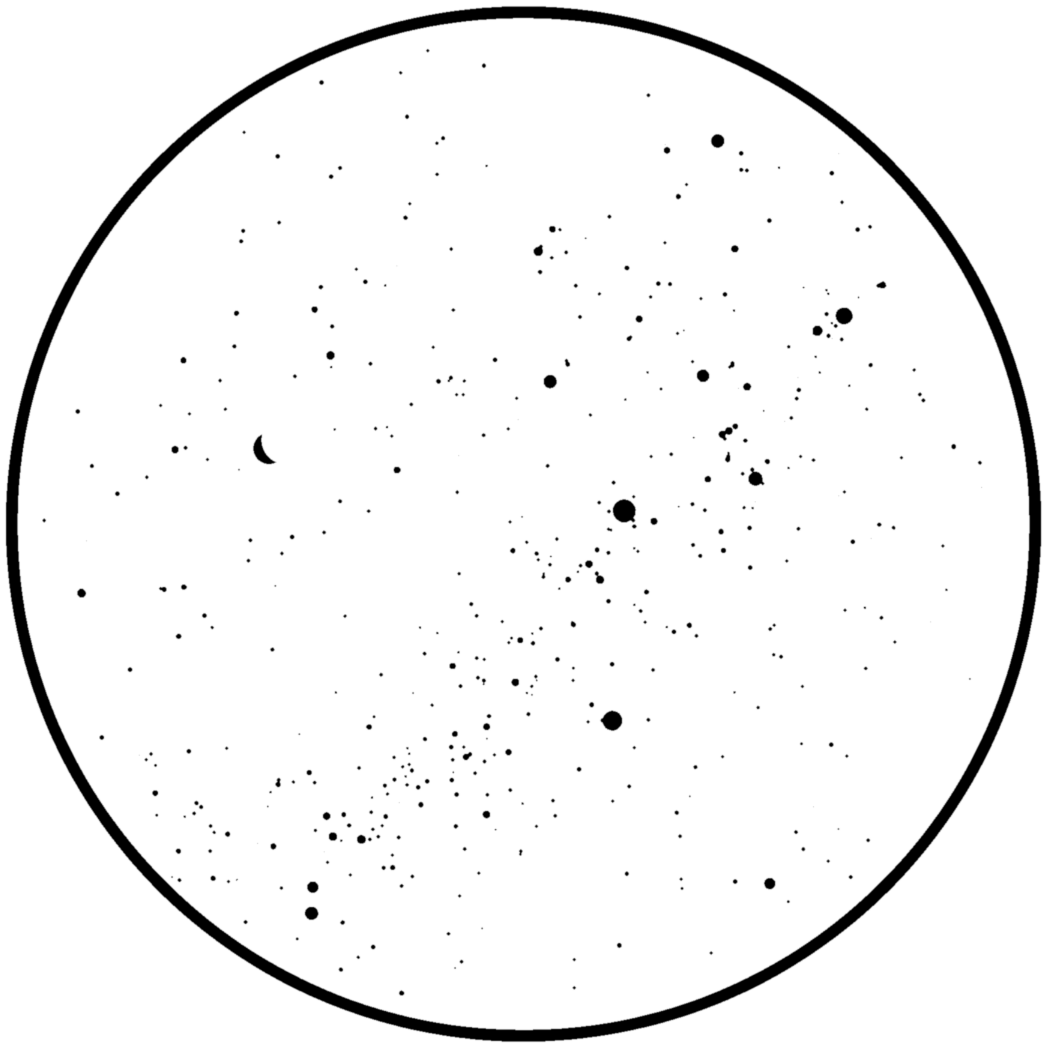
(c) The luminosity profile directly traces the mass distribution.
This is incorrect because the flat rotation curve indicates most of the mass at large radii is dark matter, not luminous. Therefore, luminosity does not accurately trace total mass.

(d) The enclosed mass within radius r grows approximately linearly with r.
This is CORRECT. From $v^2 = GM(r)/r$, constant v implies $M(r) \propto r$, consistent with the presence of a dark matter halo.

(e) The galaxy cannot contain a central supermassive black hole.
This is incorrect because the flat rotation curve describes motion at large radii and does not constrain the mass distribution in the central region; a supermassive black hole can still exist at the center.

Answer: D

The following figure is used for questions 4, 5, and 6.



You are stranded on a remote island somewhere on Earth, at an unknown date and time (in the 21st century). You look up and see the night sky.

(For avoidance of doubt, the illuminated side of the Moon is on the left.)

4. Which of the following is the best estimate of your latitude?

   (a) 40° N
   (b) 20° N
   (c) 0°
   (d) 20° S
   (e) 40° S

Solution: There are many ways to solve this, depending on your familiarity with various parts of the night sky.

From the orientation of familiar constellations such as Orion and Canis Major, we can see that north is oriented up on the image. Since Polaris is not visible and Orion (which lies on the celestial equator) is in the northern half of the sky, we can conclude that we are in the southern hemisphere.

Since the Southern Cross points towards the south celestial pole, the south celestial pole must lie at the intersection between the meridian and the line made by the Southern Cross. This is very close to the horizon, so 20° S is a better estimate than 40° S. (You can measure this with a ruler if you want to be careful.) You could also have used the fact that Canis Major is nearly directly overhead.

Answer: D

5. Which of the following is the best estimate of the month of the year?

   (a) January
   (b) April
   (c) July
   (d) September
   (e) November

Solution: The moon is roughly $\frac{1}{3}$ illuminated with the illuminated side facing east (waning), which means that the Sun is about 60° east along the ecliptic from the moon.

Since the moon is in Leo - where the Sun would be in September - this suggests the Sun is 60°, or two months, east of this point. So the current month is November.

Answer: E

6. Which of the following is the best estimate of the time of day (relative to local solar noon)?

   (a) 8:00 PM
   (b) 10:00 PM
   (c) 12:00 AM
   (d) 2:00 AM

(e) 4:00 AM

Solution:

Since the waning crescent moon is visible in the eastern sky, it must be early morning not long before sunrise. The best estimate is 4:00 AM. (At 2:00 AM the moon would still be near the horizon.)

Answer: E

Use the following information for Questions 7 and 8. One of Stephen Hawking's most famous predictions was the existence of Hawking radiation. Specifically, black holes act like thermal blackbodies and thus must emit blackbody radiation. According to Hawking, a black hole radiates a blackbody spectrum with peak wavelength λ_{max} proportional to its Schwarzschild radius R_s. Treat the surface of the blackbody as being at the black hole's event horizon.

7. According to the laws of thermodynamics and blackbody radiation, under this model, how does the power P radiated by a black hole scale with its mass M?

(a) $P \propto M^{-5}$
(b) $P \propto M^{-4}$
(c) $P \propto M^{-3}$
(d) $P \propto M^{-2}$
(e) $P \propto M^{-1}$

Solution:

Let the black hole's temperature be T. By the Wien Displacement Law,

$$T \propto \lambda_{max}^{-1} \propto R_s^{-1} = \frac{c^2}{2GM} \propto M^{-1}.$$

In this model, the Stefan-Boltzmann law thus states,

$$P = \sigma T^4 (R_s)^2 \propto (M^{-1})^4 \cdot M^2 \propto M^{-2}.$$

Answer: D

8. Next, interpret the radiated power P from the previous problem as a gradual loss of the black hole's intrinsic mass-energy $E = Mc^2$. Consider the graph of the black hole's mass versus time. Which of the following best describes the shape of the resulting graph?

(a) Increasing and concave up
(b) Constant
(c) Decreasing and concave down
(d) Linearly decreasing
(e) Decreasing and concave up

> **Solution:**
> Using the definition $P = -\frac{dE}{dt}$ and substituting our M dependences for P and E, we have $\frac{dM}{dt} \propto -M^{-2}$. Trivially, the mass decreases over time. Furthermore, as the mass decreases, the magnitude of the $-M^{-2}$ term increases, causing the slope of the graph to become more negative for larger t. The graph is thus concave down overall.
>
> **Answer: C**

9. The asteroid Pallas orbits the Sun with a semi-major axis of 2.77 AU. What is its orbital period around the Sun in years?

   (a) $(2.77)^{1/2}$
   (b) $(2.77)^{3/2}$
   (c) $(2.77)^{5/2}$
   (d) $(2.77)^{2/3}$
   (e) $(2.77)^{2/5}$

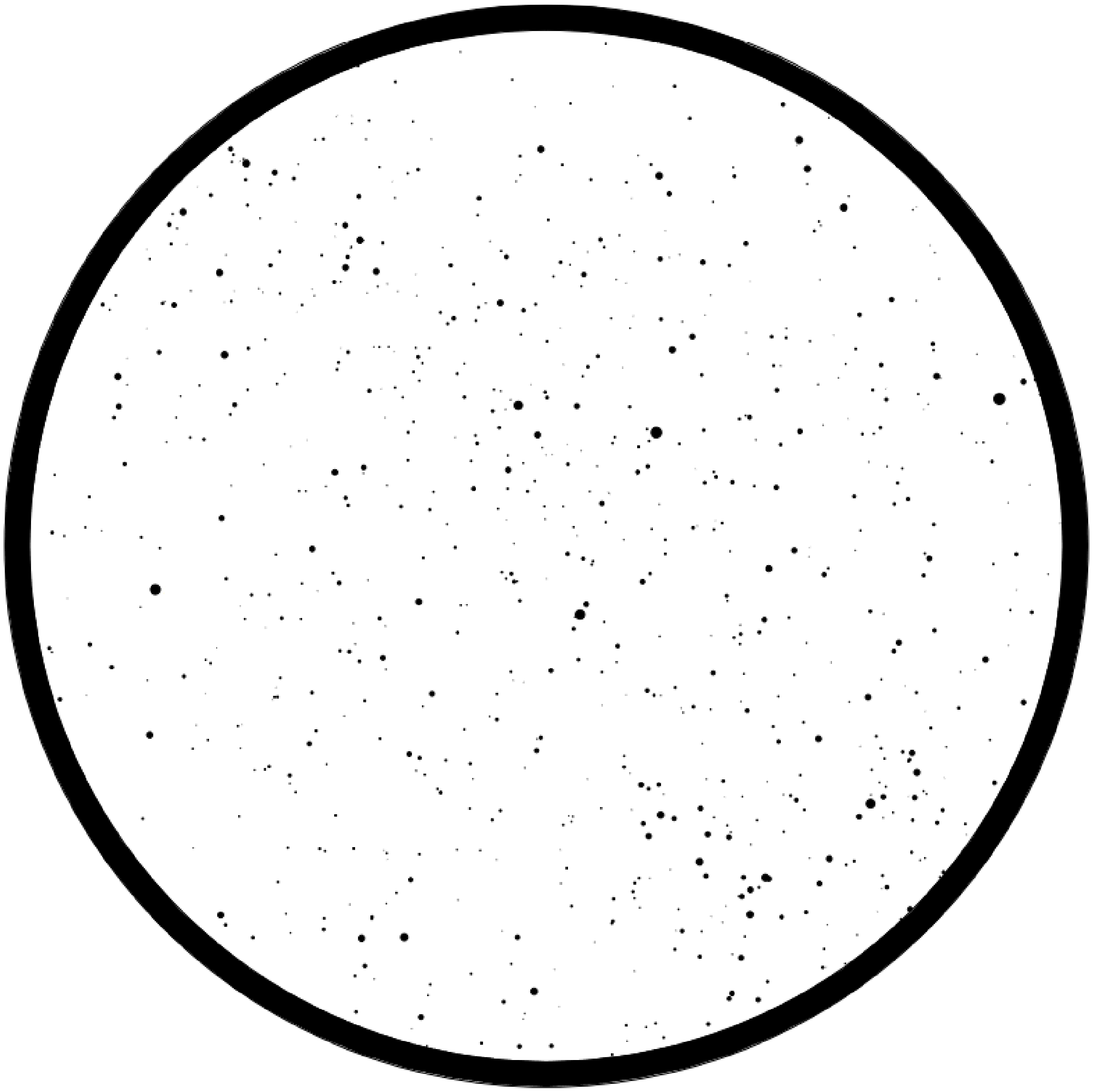
> **Solution:** When solving for the solar system, Kepler's Third law simplifies when semi-major axis (a) is expressed in AU and orbital period (P) is expressed in years, which becomes $P^2 = a^3$. Solving for P, we have $P = a^{3/2}$.
>
> **Answer: B**

10. A small satellite orbits the Earth so it stays directly above a fixed point on the equator. Using 5.97×10^{24} kg for Earth's mass and 6.37×10^6 m for its radius, how far is the satellite from the surface of the Earth? Assume its mass is 300 kg and an orbital period of 86164 s. (Find the distance above the Earth's surface, not orbital radius.)

   (a) 3.58×10^6 m
   (b) 1.56 m
   (c) 4.22×10^7 m
   (d) 3.58×10^7 m
   (e) None of the above

> **Solution:** In this scenario, gravitational force translates to centripetal force, so we equate these two. Using M for Earth's mass, m as the satellite's mass, and $T = 86400$ s for the orbital period, $\left(\frac{GMm}{r^2}\right) = m\omega^2 r$, where the angular speed is $\omega = 2\pi/T$. After substitution and rearrangement, we arrive at $r^3 = \left(\frac{GMT^2}{4\pi^2}\right)$ Plugging in the numbers, we obtain $r \sim 4.216 \times 10^7$ m, the orbital radius from Earth's center. $h = r - R = 3.58 \times 10^7$ m
>
> **Answer:D**

11. Four bright stars were erased from the sky map below. All of these stars have magnitudes lower than 3.0. Select the alternative with the four stars that were erased from the sky map.



(a) α And, α CrB, β Her, and α PsA.

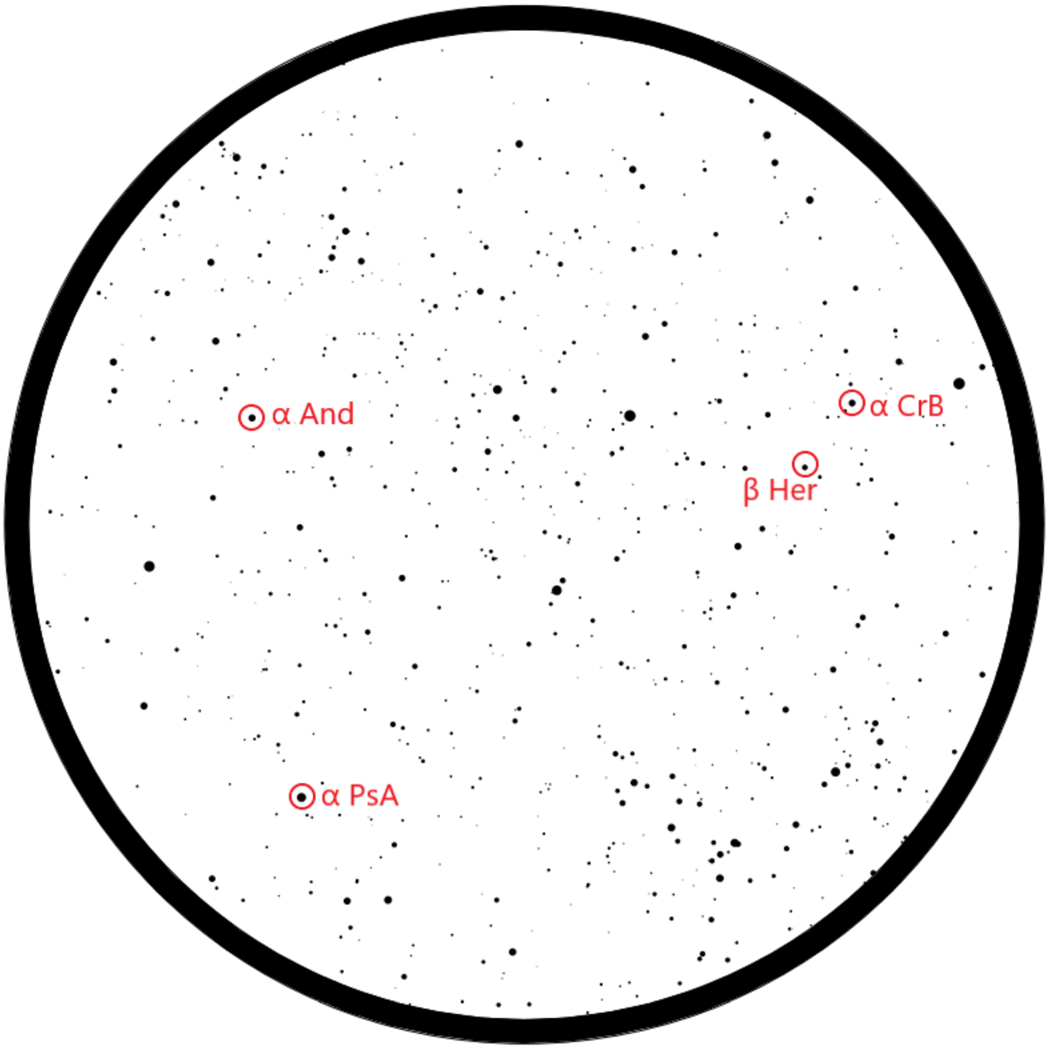
(b) α And, α Aql, β Cet, and α Lyr.

(c) α Aql, α CrB, β Her, and α Peg.

(d) β Cet, β Her, α Peg, and α Oph.

(e) α Aqr, α CrB, α Cyg, and α Her.

Solution: The four stars removed from the sky map are α And, α CrB, β Her, and α PsA.

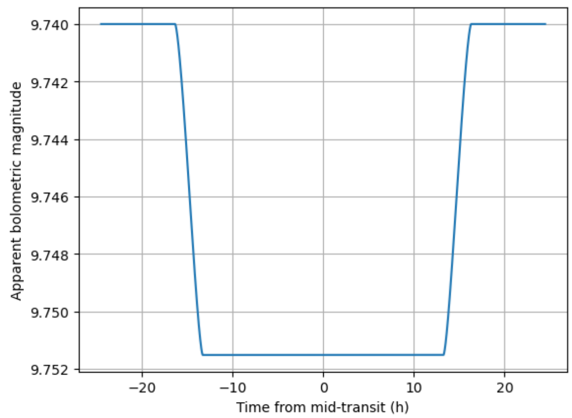


Answer: A

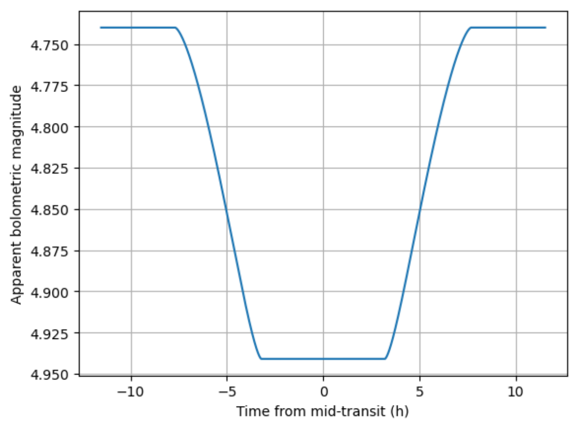
12. An extraterrestrial form of life from a distant planetary system observed a transit of Jupiter across the Sun. The transit was central, which means that the center of Jupiter's disk intersected with the center of the Sun's disk throughout the transit. Neglecting the effect of limb darkening, which of the light curves below could correspond to the transit of Jupiter seen by the extraterrestrial?

    The x-axis on the plots corresponds to the time difference in hours from the middle of the transit, and the y-axis corresponds to the apparent bolometric magnitude of the star.

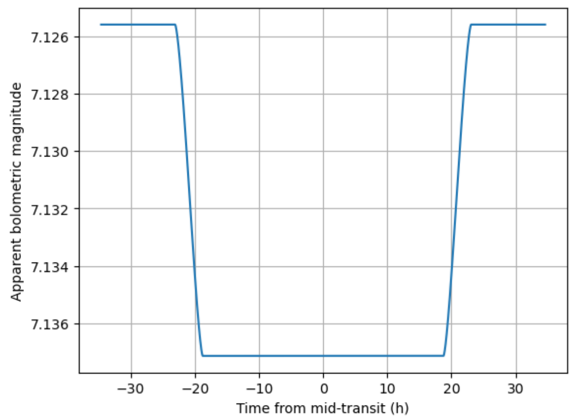
    (a)


9.740
9.742
9.744
9.746
9.748
9.750
9.752
Apparent bolometric magnitude
−20
−10
0
10
20
Time from mid-transit (h)

(b)

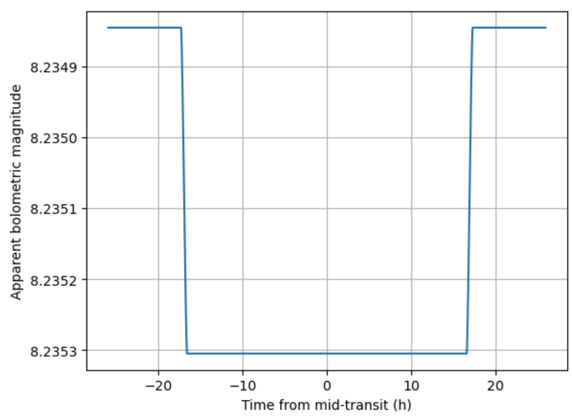

4.750
4.775
4.800
4.825
4.850
4.875
4.900
4.925
4.950
Apparent bolometric magnitude
−10
−5
0
5
10
Time from mid-transit (h)

(c)



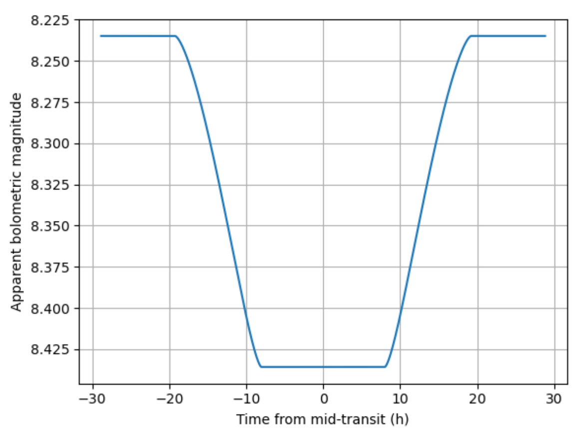
7.126
7.128
7.130
7.132
7.134
7.136
Apparent bolometric magnitude
−30
−20
−10
0
10
20
30
Time from mid-transit (h)

(d)



8.2349
8.2350
8.2351
8.2352
8.2353
Apparent bolometric magnitude
−20
−10
0
10
20
Time from mid-transit (h)

(e)



8.225
8.250
8.275
8.300
8.325
8.350
8.375
8.400
8.425
Apparent bolometric magnitude
−30
−20
−10
0
10
20
30
Time from mid-transit (h)

Solution: There a few main parameters that can be used to determine which of the light curves corresponds to the transit of Jupiter. One of them is the variation in magnitude during the transit, which is caused by the planet blocking part of the star's disk. Using Pogson's equation:

$$\begin{aligned}\Delta m &= -2.5 \log \left(\frac{\pi R_{\odot}^2 - \pi R_{♃}^2}{\pi R_{\odot}^2} \right) \\ &= -2.5 \log \left(\frac{(6.96 \times 10^8)^2 - (7.1492 \times 10^7)^2}{(6.96 \times 10^8)^2} \right) \\ &= 1.15 \times 10^{-2}\end{aligned}$$

Note that the exact magnitudes before and during the transit depend on the distance between the observer and the solar system, and only the variation in magnitude is relevant to determine the correct light curve.

Another important parameter is the duration of the transit (between the first and fourth contacts). In order to calculate that, it is important to first determine the orbital velocity of Jupiter, which can be calculated as follows assuming that the orbit is roughly circular:

$$\begin{aligned}v_{♃} &= \sqrt{\frac{GM_{\odot}}{a_{♃}}} \\ &= \sqrt{\frac{6.673 \times 10^{-11} \times 1.989 \times 10^{30}}{5.2 \times 1.496 \times 10^{11}}} \\ &= 1.3 \times 10^4 \text{ m/s}\end{aligned}$$

Therefore, the interval between the first and fourth contacts is the following:

$$\begin{aligned}\Delta t_{1-4} &= \frac{2(R_{\odot} + R_{♃})}{v_{♃}} \\ &= \frac{2(6.96 \times 10^8 + 7.1492 \times 10^7)}{1.3 \times 10^4} \\ &= 1.2 \times 10^5 \text{ s} \\ &= 33 \text{ h}\end{aligned}$$

These two parameters are already enough to determine the correct answer, but another parameter students could choose to calculate is the interval between the first and second contacts of the transit, which can be determined as follows:

$$\begin{aligned}\Delta t_{1-2} &= \frac{2R_{♃}}{v_{♃}} \\ &= \frac{2 \times 7.1492 \times 10^7}{1.3 \times 10^4} \\ &= 1.1 \times 10^4 \text{ s} \\ &= 3.0 \text{ h}\end{aligned}$$

This is the same as the interval between the third and fourth contacts. It would also be possible to use the interval between the second and third contacts as a parameter to identify the correct light curve.

Alternative A is the only light curve that has the correct variation in magnitude, transit duration, and intervals between the contacts of the transit.

Answer: A

13. What is the smallest angular separation between Algol (β Per) and the Sun throughout the year? Algol's declination is $+40°57'$, and its right ascension is 3h08min.

    (a) $11°49'$.

    (b) $15°45'$.

    (c) $22°26'$.

    (d) $26°56'$.

    (e) $29°03'$.

Solution: The Sun moves along the ecliptic throughout the year. Therefore, the minimum angular separation between the Sun and Algol corresponds simply to Algol's ecliptic latitude. The figure below shows the spherical triangles that can be used to obtain the coordinate conversion formulas between the equatorial coordinates and the ecliptic coordinates.

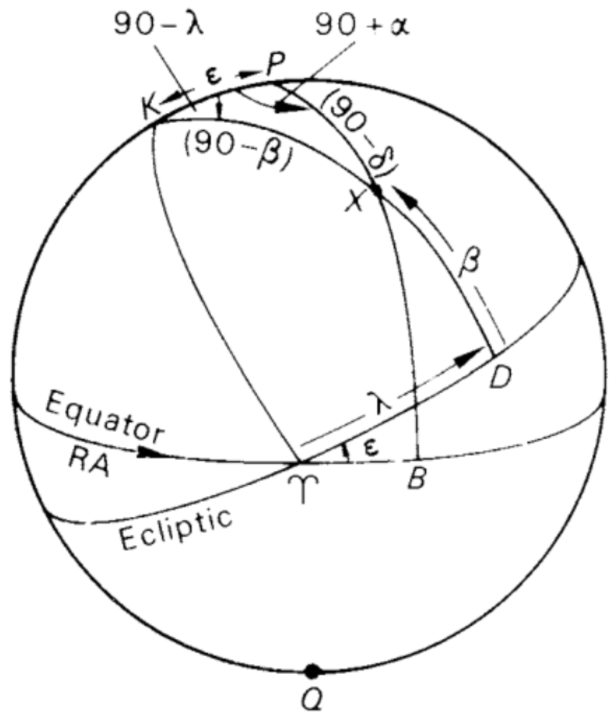


Figure 1: *
Source of the image: Astronomy Principles and Practice by Roy and Clarke

In the figure, α denotes the right ascension, δ denotes the declination, λ denotes the ecliptic longitude, β denotes the ecliptic latitude, and ε denotes the obliquity of the ecliptic.

Using the spherical law of cosines on the PKX triangle:

$$\begin{aligned}
\cos(90^\circ - \beta) &= \cos(90^\circ - \delta)\cos(\varepsilon) + \sin(90^\circ - \delta)\sin(\varepsilon)\cos(90^\circ + \alpha) \\
\sin(\beta) &= \sin(\delta)\cos(\varepsilon) - \cos(\delta)\sin(\varepsilon)\sin(\alpha) \\
&= \sin(40^\circ 57')\cos(23.44^\circ) - \cos(40^\circ 57')\sin(23.44^\circ)\sin(3\text{h}08\text{min}) \\
&= 0.3816 \\
\beta &= 0.3886 \\
\beta &= 22.43^\circ \\
&= 22^\circ 26'
\end{aligned}$$

Answer: C

14. Consider a cloud of hot gas with radius R and total (electromagnetic) power emission rate P. Keeping the average temperature of gas particles and average emissivity per particle constant, the cloud now expands to a radius $2R$. How does the net total power P radiated away, observed from far away, change in the cases where the cloud is optically thin (optical depth $\tau \ll 1$) and

optically thick (optical depth $\tau \gg 1$), respectively? Assume the total number of particles remains constant.

(a) Remains approximately constant, remains approximately constant

(b) Remains approximately constant, increases significantly

(c) Increases significantly, decreases significantly

(d) Increases significantly, remains approximately constant

(e) Increases significantly, increases significantly

Solution: If the gas is optically thin, then light emitted from one particle has a negligible probability of being absorbed and/or scattered by another particle, meaning that the output power will be essentially equal to the total power of all the particles both before and after, meaning that it remains approximately constant. (Note that because of Kirchoff's laws, this curve may likely deviate significantly from that of a perfect blackbody.)

If the gas is optically thick, then by Kirchoff's laws, we essentially have a blackbody emission curve. By the Stefan-Boltzmann Law, the power emitted is just $P = \sigma(4\pi r^2)\epsilon T^4$. Since r doubles throughout this process, the power P increases by a factor of 4.

Answer: B

15. Lunar laser ranging experiments show that, due to tidal effects, the semi-major axis of the Moon's orbit increases by around 38 mm per year. Assuming that the recession rate is constant, by approximately how much does this recession increase the Moon's sidereal orbital period each year? The mass of the Moon is 7.3×10^{22} kg. *Hint:* $(1+x)^\beta \approx 1 + \beta x$ *for* $|x| \ll 1$.

(a) 230 μs

(b) 350 μs

(c) 380 μs

(d) 690 μs

(e) 230 ms

Solution:

Note that there are multiple ways to solve this problem. By Kepler's Third Law,

$$T_{sid} = \sqrt{\frac{4\pi^2 a^3}{G(M_\oplus + M_{☾})}}.$$

Taking $a \to a + \delta a$,

$$\begin{aligned}\delta T_{sid} &= \sqrt{\frac{4\pi^2(a+\delta a)^3}{G(M_\oplus + M_☾)}} - \sqrt{\frac{4\pi^2 a^3}{G(M_\oplus + M_☾)}} \\ &= \sqrt{\frac{4\pi^2 a^3}{G(M_\oplus + M_☾)}}\left((1+\delta a/a)^{3/2} - 1\right) \approx \frac{3}{2}\frac{\delta a}{a}T_{sid} \\ &= \frac{3}{2}\frac{38\times10^{-3}\,\mathrm{m}}{3.84399\times10^{8}\,\mathrm{m}}\sqrt{\frac{4\pi^2(3.84399\times10^8\,\mathrm{m})^3}{(6.673\times10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})(5.976\times10^{24}\,\mathrm{kg} + 7.3\times10^{22}\,\mathrm{kg})}} \\ &= 350\,\mu\mathrm{s}.\end{aligned}$$

Answer: B

16. Kepler-22b is an exoplanet discovered by transiting with an orbital period of 290 days. Suppose it has a mass of 4.84×10^{25} kg and a moon similar to Earth's, with a prograde orbit near the ecliptic plane of the Kepler-22 system and a semi-major axis of 7.70×10^8 m. What is the length of a synodic lunar month, i.e., the time between successive full moons, for an observer on Kepler-22b? You can neglect the mass of the moon.

    (a) 25.0 days
    (b) 27.3 days
    (c) 29.5 days
    (d) 30.2 days
    (e) 33.6 days

Solution:
First, use Kepler's Third Law to find the sidereal period: $GMT_{sid}^2 = 4\pi^2 a^3$, so

$$T_{sid} = \sqrt{\frac{4\pi^2 a^3}{GM}} = \sqrt{\frac{4\pi^2(7.70\times10^8\,\mathrm{m})^3}{(6.673\times10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})(4.84\times10^{25}\,\mathrm{kg})}}\cdot\frac{1\,\mathrm{day}}{24\cdot3600\,\mathrm{s}} = 27.341\,\mathrm{days}.$$

Then, use the relative rate of motion of the moon around the planet and the planet around the star to find the synodic period: $1/T_{syn} = 1/T_{sid} - 1/T_p$, so

$$T_{syn} = \left(\frac{1}{27.341\,\mathrm{days}} - \frac{1}{290\,\mathrm{days}}\right)^{-1} = 30.2\,\mathrm{days}.$$

Answer: D

17. A quickly-moving black hole passes through the center of a large star cluster. While the black hole does not experience any direct collisions with any stars in the cluster or significantly disrupts any stellar atmospheres, the black hole imparts a fraction of its momentum to several stars in the cluster close to the black hole's trajectory. (This is known as dynamic gravitational friction and can be explained purely with Newtonian mechanics.) Some time passes, and due to complex

multi-body interactions, the additional energy and momentum get distributed over the entire star cluster.

After a sufficiently long time, which of the following statements are likely TRUE regarding the system (compared to before the black hole event), assuming no further perturbations occur?

(I) The average peak blackbody emission wavelength among stars in the cluster decreases significantly.

(II) Stars are on average further apart from each other than they were before.

(III) Stars have lower relative velocities with respect to each other than they had before.

(a) I only
(b) II only
(c) III only
(d) II and III
(e) I and II

Solution: The black hole initially increases the (relative) velocities of several stars in the cluster. This has the effect of increasing the total average energy $\langle E \rangle$ of stars in the cluster. However, after a long time, this energy gets distributed over the entire cluster. By the virial theorem, this causes the average potential energy $\langle U \rangle$ to increase (making stars further apart, so (II) is true) as well as causing the average kinetic energy $\langle K \rangle$ to **decrease**, making (III) counterintuitively correct. (I) is trivially false because the additional energy manifests as a change in bulk kinetic energy of entire stars as opposed to their internal thermal energies, meaning that the spectra are essentially unchanged.

Answer: D

18. The most distant known galaxy, MoM-z14, was recently observed by the James Webb Space Telescope and has a redshift of $z = 14.44$. If the current cosmic microwave background (CMB) temperature is 2.725 K, what was the CMB temperature at MoM-z14?

(a) 0.1887 K
(b) 39.35 K
(c) 42.07 K
(d) 568.2 K
(e) 3208 K

Solution:

You can either directly use the relationship between temperature and scale factor or use Wien's law. Redshift $z = \Delta\lambda/\lambda_0 = \lambda/\lambda_0 - 1$, so the original wavelength was shorter by a factor of $\lambda_0/\lambda = 1/(1+z)$. According to Wien's law, λT is constant, so

$$T = \frac{\lambda T_{now}}{\lambda_0} = (1+z)T_{now} = (1+14.44)\cdot 2.725\,\text{K} = 42.07\,\text{K}.$$

Answer: C

19. Assume a spherical star has a density profile described by the equation below:

$$\rho(r) = \left(1 - \frac{r^3}{R^3}\right) \rho_c, \quad 0 \leqslant r \leqslant R$$

Find ρ_c in terms of the total mass, M, and the radius of the star, R.

(a) $\frac{3M}{\pi R^3}$
(b) $\frac{15M}{8\pi R^3}$
(c) $\frac{3M}{2\pi R^3}$
(d) $\frac{21M}{16\pi R^3}$
(e) $\frac{6M}{5\pi R^3}$

Solution: First, find ρ_c in M:

$$dm = 4\pi r^2 \rho(r)\ dr$$
$$\int_0^M dm = \int_0^R 4\pi r^2 \left(1 - \frac{r^3}{R^3}\right) \rho_c\ dr$$
$$M = 4\pi\rho_c \left[\frac{r^3}{3} - \frac{r^6}{6R^3}\right]_0^R = \frac{2\pi}{3} R^3 \rho_c$$
$$\rho_c = \boxed{\frac{3M}{2\pi R^3}}$$

Answer: C

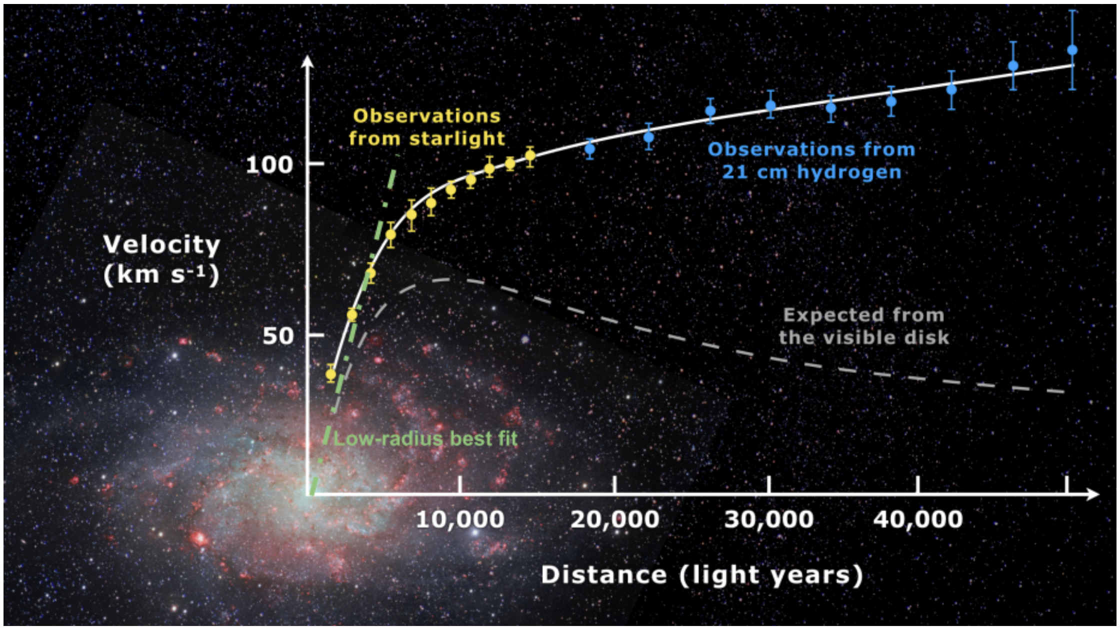
20. Some stars formed in the early universe are still observable today. If an observer observed one and it is still in the main sequence phase, what would most likely be its mass?

(a) < 2 $M_\odot$
(b) $2 - 8$ $M_\odot$
(c) $8 - 12$ $M_\odot$
(d) $12 - 20$ $M_\odot$
(e) > 20 $M_\odot$

Solution: Stars formed in the early universe that remain observable today must have very long main-sequence lifetimes. Stellar lifetime decreases with increasing mass as they rapidly burn their fuel. Hence, stars with masses above a few solar masses live for only millions to hundreds of millions of years. In contrast, low-mass stars $\lesssim 1$ $M_\odot$ can live for tens of billions of years This means, any main sequence stars formed in the early universe that still exist today must be low-mass, or the answer is **(A)**. You may explore the stellar mass–lifetime relationship using the interactive simulation at https://foothillastrosims.github.io/stellar-lifespan/.

Answer: A

Use the following information for Questions 21, 22, and 23. Consider the following graph, which shows the mean orbital velocity of stars around a galaxy's center as a function of distance from the center. The graph also shows the expected orbital velocities (dashed line) from observations of visible matter. The discrepancy between the observed and expected values is assumed to be due to invisible dark matter.



21. Consider orbital radii within the most luminous part of the disk, at around $r \leqslant 5000$ ly. Let $M(r)$ be the total mass (of all forms) enclosed within a distance r from the galactic center. Approximating the rotation curve by the low-radius best fit (dot-dash) line, we can infer that $M(r)$ is roughly proportional to r^n for some value of n (over small r).

    In light of the value of n, and for $r \leqslant 5000$ ly, which of the following hypotheses is **MOST** consistent with the data?

    (a) Near the galactic center, the mass is concentrated in and is nearly uniform over the 2D galactic plane.

    (b) The majority of the contribution to $M(r)$ comes from the existence of a supermassive black hole at the galactic center.

    (c) Dark energy-based repulsion from the galactic center induces a mass distribution of the form $\rho(r) \propto r^2$.

    (d) Complex gravitational interactions with the dark matter halo make such power laws impossible, with no single value of n being consistent with the best fit line.

    (e) Gas, dust, and stars are approximately uniformly distributed in 3D space near the galactic center with no significant radius dependence of density.

Solution:

By Newton's Law of Gravity, considering an object of mass m,

$$G\frac{M(r)m}{r^2} = \frac{mv^2}{r} \longrightarrow M(r) \propto v^2 r$$

According to the graph, we have $v(r) \propto r$ for small values of r, so $M(r) \propto r^3$. The corresponding values of n for the answer choices are as follows:

(A): $n = 2$, (B): $n = 0$, (C): $n = 5$, (D): no such n, (E): $n = 3$.

Answer: E

22. Now, for larger values of r, the discrepancy between the observed and expected rotation curves begins to appear more significant. Assume that the expected rotation curve (dashed gray) curve is calculated off the distribution of all baryonic ("standard") matter, while the observed rotation curve (solid white) is due to the combination of baryonic matter and dark matter. Within a radius of $r = 40,000$ ly from the galactic center, and according to the graph, which of the following is closest to the ratio between the dark matter mass and the baryonic matter mass enclosed?

    (a) 2
    (b) 5
    (c) 10
    (d) 25
    (e) 50

Solution: Using the same equation as before, we get $M(r) \propto v(r)^2$. It is relatively clear that the value of the true velocity curve is around 3.5 larger than that of the expected velocity curve, meaning that the answer is around $3.5^2 - 1 \approx 11$, giving answer choice C. This is the only answer choice within the ballpark of any reasonable estimation.

[Note: measuring the two heights with a ruler gives a more precise estimate of $(12.0/3.45)^2 - 1 \approx 11.10$, which is reasonably close.]

Answer: C

23. Consider the following three assertions related to the techniques used to measure the overall rotation curve, in the context of the aforementioned figure:

    (I) Interstellar dust is colder at lower values of r, making the dust obscure less starlight.

    (II) O/B type stars in the spiral arms of galaxies emit significant quantities of UV light, ionizing nearby H-I clouds and thus inhibiting the emission of 21-cm radiation in their vicinities.

    (III) The lower star number density for higher values of r means that the stars that do exist typically move at relativistic speeds, complicating Doppler-based stellar velocity measurements.

    Which of the above are **TRUE**?

    (a) I only

(b) II only

(c) III only

(d) I and II only

(e) None of the above

Solution:

(I) is false because interstellar dust is hotter for lower values of r, since the dust that exists is closer to more stars (that are also hotter on average), making the dust hotter overall.

(II) is true and is a direct consequence of the extremely long decay time of the hyperfine hydrogen excited state.

(III) is false because stars orbit at nowhere near relativistic speeds, as can be seen from the fact that the rotational velocity is within an order of magnitude of 100 km/s everywhere, much less than the speed of light. The correct answer is II only.

Answer: B

24. Suppose the Honeyhive galaxy and the Gold Leaf galaxy are diametrically opposite when viewed from the Earth, and the following sequence of events happens:

(a) A type Ia supernova happens in the Honeyhive galaxy.

(b) Astronomers on Earth measure the redshift of (the spectral lines) in the Honeyhive supernova to be $z = 0.5$. At the same time, another supernova happens in the Milky Way galaxy.

(c) Aliens in the Gold Leaf galaxy measure the redshifts of the the supernovae in the Honeyhive and Milky Way galaxies.

If the alien astronomers measure the redshift of the Milky Way supernova to be $z = 0.8$, what redshift would they measure for the Honeyhive supernova?

You may assume that the local velocities of the galaxies are negligible, that the universe is simple homogeneous, and isotropically expanding, and that redshift depends only on the scale factor and is the same function of distance for all observers.

(a) 0.2

(b) 0.4

(c) 1.3

(d) 1.7

(e) 2.7

Solution: Suppose some known spectral line in the Honeyhive galaxy supernova has wavelength λ. By the definition of redshift, this spectral line has wavelength $(1 + z)\lambda = 1.5\lambda$ when it reaches Earth. Similarly, any light emitted from the supernova in the Milky Way with have its wavelength multiplied by 1.8 when it reaches the Gold Leaf galaxy.

Since the light from the Honeyhive galaxy supernova passes through the Milky Way and starts heading towards the Gold Leaf galaxy at the same time as the Milky Way supernova, it will be redshifted by the same amount. Thus, any spectral line with wavelength λ at the time it leaves the Honeyhive galaxy will have wavelength $1.8 \cdot (1.5\lambda) = 2.7\lambda$, meaning the measured redshift for the Honeyhive galaxy will be $2.7 - 1 = 1.7$.

Answer: D

25. The Balmer-α, H-I, and Lyman-α lines are important spectral lines in astronomy, all from hydrogen.

- The Balmer-α line occurs when an electron in a hydrogen atom goes from the $n = 3$ state to $n = 2$.
- The H-I line occurs from an electron in the 1s shell of a hydrogen atom undergoing a spin-flip transition between two hyperfine levels.
- The Lyman-α line is the emission line that occurs when an electron in a hydrogen atom goes from $n = 2$ to $n = 1$.

Order these lines from the *shortest* to *longest* wavelength.

(a) Balmer-α, H-I, Lyman-α
(b) H-I, Balmer-α, Lyman-α
(c) Lyman-α, H-I, Balmer-α
(d) Lyman-α, Balmer-α, H-I
(e) Balmer-α, Lyman-α, H-I

Solution: Without prior knowledge of these lines, one can determine that the Lyman-α line has a shorter wavelength than the Balmer-α line using the Rydberg equation:

$$\frac{1}{\lambda} = R_H \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right).$$

Since $\frac{1}{1^2} - \frac{1}{2^2}$ is greater than $\frac{1}{2^2} - \frac{1}{3^2}$, the transition from $n = 2$ to $n = 1$ will have a smaller wavelength than $n = 3$ to $n = 2$.

The H-I line comes from an energy difference between two hyperfine levels in the same orbital, while the Lyman-α and Balmer-α lines come from transitions between two orbitals, so the H-I line will have less energy, and thus a longer wavelength, than both of the other lines.

Of course, one can also determine the answer directly if they already know which parts of the electromagnetic spectrum each line is in (Lyman-α is ultraviolet, Balmer-α is visible, and H-I is in the microwave range).

Answer: D

26. Star X and star Y form an eclipsing binary system, where star X is larger, and fully blocks star Y during an eclipse. During the primary eclipse, the apparent magnitude of the binary system increases by 0.200, and during the secondary eclipse, the apparent magnitude only increases by 0.100. If T_X is the surface temperature of star X and T_Y is the surface temperature of star Y, which of the following is a possible value of T_X/T_Y? Ignore limb darkening.

(a) 0.679
(b) 1.000
(c) 1.176
(d) 1.414
(e) 2.000

Solution: Since star X is larger, then during the eclipses either star Y is fully blocked (when star X is in front), or a portion of star X equal in area to star Y is blocked (when star Y is in front).

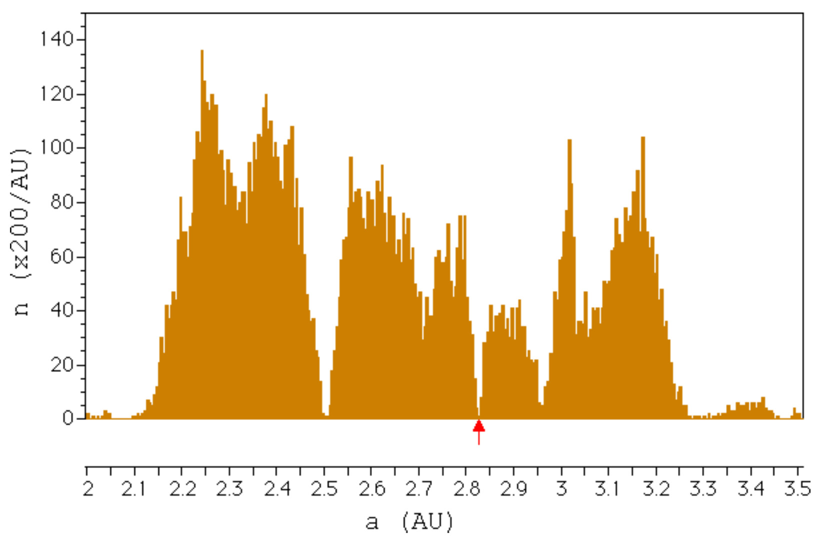
During the primary eclipse, the fraction of light blocked is $1 - 10^{-0.4 \cdot 0.200} = 0.168$, and during the secondary eclipse, the fraction of light blocked is $1 - 10^{-0.4 \cdot 0.100} = 0.0880$.

Since the areas eclipsed are the same, we know that the ratio of the surface brightness of the brighter star to the dimmer star is 0.168/0.0880, so the ratio of surface temperature of the brighter star to the dimmer star is $(0.168/0.0880)^{1/4}$.

Since we don't know which of star X and star Y is brighter, T_X/T_Y can be either $\sqrt[4]{\frac{0.168}{0.0880}} = 1.176$ or $\sqrt[4]{\frac{0.0880}{0.168}} = 0.850$, of which the first value is an answer choice.

Answer: C

27. When plotting the distribution of asteroids in the asteroid belt by their semi-major axis, we can see several regions, known as "Kirkwood gaps," where asteroids' orbits are unstable due to an orbital resonance with Jupiter ($a = 5.20$ AU). What orbital resonance is responsible for the gap marked with the arrow?



(a) 8:5
(b) 11:6
(c) 9:4
(d) 5:2

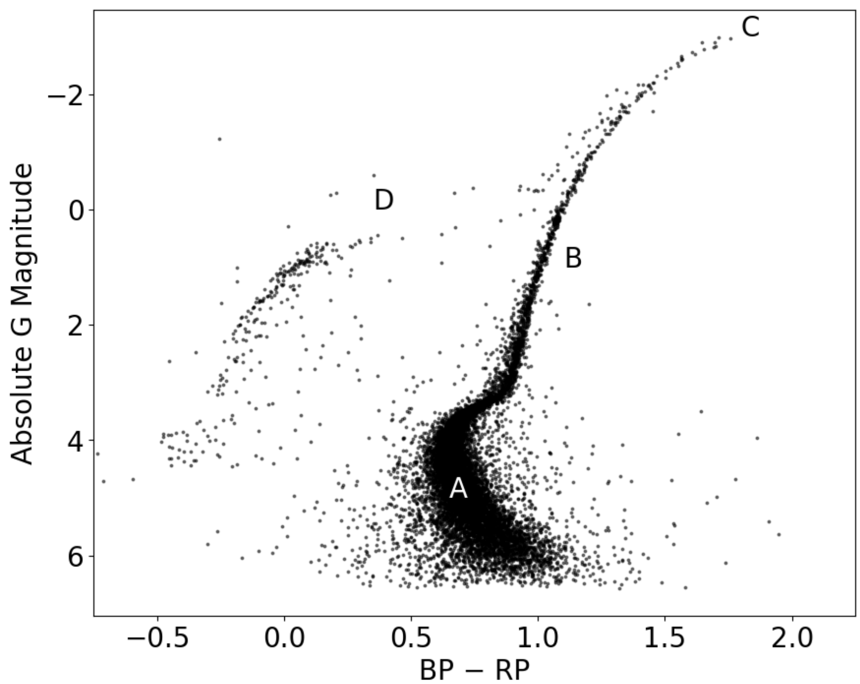
(e) 3:1

Solution:

The red arrow is at a semi-major axis of approximately 2.82 AU. By Kepler's 3rd law, the ratio of the orbital period of an asteroid with this semi-major axis to Jupiter's orbital period is $(5.20/2.82)^{3/2} = 2.50$. This is a 5:2 ratio.

Answer: D

28. Below is an HR diagram for a globular cluster from Gaia DR3 data:



The figure has labeled regions: "A", "B", "C", and "D". 1) Identify the region where stars undergo hydrogen burning in their cores. 2) Identify the region where stars experience *Helium flash*.

(a) 1. Region A; 2. Region B
(b) 1. Region A; 2. Region C
(c) 1. Region B; 2. Region C
(d) 1. Region B; 2. Region D
(e) 1. Region C; 2. Region D

Solution: For the labeled regions:

(1) Region A is the Main Sequence phase, where stars undergo hydrogen fusion in their core.

(2) Region B is the Red Giant Branch, where stars undergo hydrogen burning in their shell. (No burning in their core)

(3) Region C is the tip of the Red Giant Branch, and for low-mass stars, they typically experience *Helium flash* here. The *Helium flash* indicates the first time of Helium burning in the core of low-mass stars.

(4) Region D is the Horizontal Branch, where stars undergo helium fusion in their core and hydrogen shell burning.

So, region A and C is the most appropriate.

Answer: B

29. Which of the following is closest to the ratio of the number of photons from the Sun that strike the Earth each second to the number of photons from the Cosmic Microwave Background that strike the Earth each second? *(Assume the Cosmic Microwave Background to be a uniform and isotropic blackbody signal at $T_c \approx 2.725\,\mathrm{K}$. As a hint, you may approximate all photons from a given source as being at the peak emission energy.)*

(a) 5×10^4
(b) 2×10^5
(c) 3×10^7
(d) 1×10^8
(e) 4×10^8

Solution: Notice that if $L_\odot$ is the power radiated by the Sun, then the total solar power incident on the Earth is $P'_\odot = \frac{\pi R_\oplus^2}{4\pi D^2} L_\odot$, where D is the Sun-Earth distance. On the other hand, the power incident on the Earth by CMB photons is the same power as if Earth was radiating at a temperature T_c (because then the Earth would be in thermal equilibrium with the CMB and there would be no net thermal exchange), so the total CMB power incident on the Earth is $P'_c = (4\pi R_\oplus^2)\sigma T_c^4$. Lastly, notice that by the Wien Displacement Law, the mean/typical wavelength emitted are proportional to T^{-1}, meaning that the average photon energy is proportional to T, and so if N is the numerical photon emission rate, then $P \propto NT \longrightarrow N \propto P/T$. The desired ratio is thus,

$$\frac{P'_\odot}{P'_c} \cdot \frac{T_c}{T_\odot} = \frac{L_\odot}{16\pi D^2 T_c^3 T_\odot} \approx \frac{(3.827 \times 10^{26}\ \mathrm{W})}{16\pi (1.496 \times 10^{11}\ \mathrm{m})^2 (5.67 \times 10^{-8}\ \frac{\mathrm{W}}{\mathrm{m^2 \cdot K^4}})(2.725\ \mathrm{K})^3 (5770\ \mathrm{K})},$$

which is 51400 to three significant figures. Alternatively, using the formula $L_\odot = 4\pi\sigma R_\odot^2 T_\odot^4$ gives the desired ratio as

$$\left(\frac{T_\odot}{T_c}\right)^3 \left(\frac{R_\odot^2}{4D^2}\right),$$

which gives the same numerical result to within 3 significant figures. The factor of $R_\odot^2/4D^2$ in the latter solution can be interpreted as comparing the "fractions" of the sky taken up by each source. Additionally, it helps to see that the Earth poses an effective cross section of $\pi R_\oplus^2$ to solar radiation but all $4\pi R_\oplus^2$ of the Earth's surface area uniformly receives CMB radiation, with the resulting factor of $\pi R_\oplus^2/4\pi R_\oplus^2 = 1/4$ being another way of intuiting the 4 in the denominator of our second final expression.

The numerical answer (from either method) is closest to 5×10^4,

Answer: A

30. Consider two galaxies:

- Galaxy S is a spiral galaxy with blue spiral arms.
- Galaxy E is an elliptical galaxy with no blue stars, only red stars.

Which of the following statements are likely TRUE?

(I) We can find planetary nebulae in both galaxies.
(II) Galaxy S is likely to have ongoing star formation.
(III) Galaxy E is likely to have ongoing star formation.
(IV) Type II supernovae are more likely to be found in Galaxy S
(V) Type Ia supernovae are likely to be found only in Galaxy E

(a) (I), (II), (III)
(b) (I), (II), (IV)
(c) (I), (III), (IV)
(d) (I), (IV), (V)
(e) (II), (IV), (V)

Solution: From the colors of the galaxies, Galaxy S (a spiral galaxy with blue spiral arms) contains a younger stellar population, while Galaxy E (an elliptical galaxy with no blue stars, only red stars) contains an older stellar population. From this, we can know:

(I) **True**. Planetary nebulae come from low/intermediate-mass stars, which exist in both young and old populations. Thus, we can find planetary nebulae in both galaxies.

(II) **True**. Blue spiral arms indicate the presence of short-lived, massive stars with a lot of gas around. This implies there is likely an ongoing star formation.

(III) **False**. Galaxy E consists primarily of old, red stars and generally lacks cold gas, making ongoing star formation unlikely.

(IV) **True**. Type II supernovae are more likely to be found in young populations as their blue, massive stars die quickly. Thus, Galaxy S is more likely to have a Type II supernova.

(V) **False**. Type Ia supernovae usually originate from white dwarfs in binary systems and can occur in both young and old stellar populations, so they are not exclusive to Galaxy E.

Answer: B